

# Mean curvature flow of spacelike graphs

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## Abstract

We prove the mean curvature flow of a spacelike graph in  $(\Sigma_1 \times \Sigma_2, g_1 - g_2)$  of a map  $f : \Sigma_1 \rightarrow \Sigma_2$  from a closed Riemannian manifold  $(\Sigma_1, g_1)$  with  $\text{Ricci}_1 > 0$  to a complete Riemannian manifold  $(\Sigma_2, g_2)$  with bounded curvature tensor and derivatives, and with sectional curvatures satisfying  $K_2 \leq K_1$ , remains a spacelike graph, exists for all time, and converges to a slice at infinity. We also show, with no need of the assumption  $K_2 \leq K_1$ , that if  $K_1 > 0$ , or if  $\text{Ricci}_1 > 0$  and  $K_2 \leq -c$ ,  $c > 0$  constant, any map  $f : \Sigma_1 \rightarrow \Sigma_2$  is trivially homotopic provided  $f^*g_2 < \rho g_1$  where  $\rho = \min_{\Sigma_1} K_1 / \sup_{\Sigma_2} K_2^+ \geq 0$ , in case  $K_1 > 0$ , and  $\rho = +\infty$  in case  $K_2 \leq 0$ . This largely extends some known results for  $K_i$  constant and  $\Sigma_2$  compact, obtained using the Riemannian structure of  $\Sigma_1 \times \Sigma_2$ , and also shows how regularity theory on the mean curvature flow is simpler and more natural in pseudo-Riemannian setting than in the Riemannian one.

## 1 Introduction

Let  $M$  be a smooth manifold of dimension  $m \geq 2$ , and  $F_0 : M \rightarrow \bar{M}$  a smooth submanifold immersed into a  $(m+n)$ -dimensional Riemannian or pseudo-Riemannian

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**MSC 2000:** Primary:53C21, 53C40 Secondary: 58D25, 35K55.

**Key Words:** mean curvature flow, spacelike submanifold, maximum principle, homotopic maps.

<sup>†</sup> Partially supported by NSFC (No.10501011) and by Fundação Ciência e Tecnologia (FCT) through a FCT fellowship SFRH/BPD/26554/2006.

<sup>‡</sup> Partially supported by FCT through the Plurianual of CFIF and POCI-PPCDT/MAT/60671/2004.

manifold  $(\bar{M}, \bar{g})$ . The mean curvature flow is a smooth family of immersions  $F_t = F(\cdot, t) : M \rightarrow \bar{M}$  evolving according to

$$\begin{cases} \frac{d}{dt}F(x, t) &= H(x, t), \quad x \in M, \\ F(\cdot, 0) &= F_0, \end{cases} \quad (1.1)$$

where  $H$  is the mean curvature vector of  $M_t = (M, g_t = F_t^* \bar{g}) = F_t(M)$ .

The mean curvature flow of hypersurfaces (i.e. (1.1) with  $n = 1$ ) in a Riemannian manifold has been extensively studied in the last two decades. Recently, mean curvature flow of submanifolds with higher co-dimensions has been paid more attention, see [4, 18, 20, 21, 24] for example. In [21], the graph mean curvature flow is studied in Riemannian product manifolds, and it is proved long-time existence and convergence of the flow under suitable conditions.

When  $\bar{M}$  is a pseudo-Riemannian manifold, (1.1) is the mean curvature flow of spacelike submanifolds. This flow for spacelike hypersurfaces has also been strongly studied, see [1, 5, 6, 8] and references therein. To our knowledge, very little is known on mean curvature flow in higher codimensions except in a flat space  $\mathbb{R}_n^{n+m}$  [24]. In this paper, we partially apply Wang's approach [21] of using the mean curvature flow in a Riemannian product space to deform a map between Riemannian manifolds, but we use the pseudo-Riemannian structure of the product. As a result we obtain a reformulated and largely extended version of the main results in [21] and most of [19], to non constant sectional curvatures  $K_i$ , and applied to a set of maps satisfying a less restrictive condition, after using a simple argument of rescaling the metric in the target space  $\Sigma_2$ , in a convenient way. The pseudo-Riemannian structure turns out to give a simpler and more natural tool to provide an existence result on the deformation of a map to a totally geodesic one or to a constant one by some curvature flow under quite weak curvature conditions. In [21] it is necessary to use White's regularity theorem [23], where a monotonicity formula due to Huisken [12] plays a fundamental role, to detect possible singularities of the mean curvature flow, while in pseudo-Riemannian case, because of good signature in the evolution equations, we have better regularity, and therefore we require fewer restrictions on the curvatures of  $\Sigma_1$  and  $\Sigma_2$  in the main theorem 1.1 as well on the map  $f$  itself in theorem 1.2. We also believe that this "pseudo-Riemannian" trick may be applied to some other geometric evolution equations to obtain the convergence of the flow in a more efficient way.

Let  $\bar{M} = \Sigma_1 \times \Sigma_2$  be a product manifold of two Riemannian manifolds  $(\Sigma_1, g_1)$  of dimension  $m$  and  $(\Sigma_2, g_2)$  of dimension  $n$ , with the pseudo-Riemannian metric  $\bar{g} = g_1 - g_2$ , where  $\Sigma_i, i = 1, 2$  has sectional curvature  $K_i$  and Ricci tensor  $Ricci_i$ .

Assume  $M$  is a spacelike graph

$$M = \Gamma_f = \{(p, f(p)) : p \in \Sigma_1\}$$

of a smooth map  $f : \Sigma_1 \rightarrow \Sigma_2$ , with induced metric  $g$ . The graph map,  $\Gamma_f : \Sigma_1 \rightarrow M$ ,  $\Gamma_f(p) = (p, f(p))$ , identifies isometrically  $(M, g)$  with  $\Sigma_1$  with the graph metric  $g_1 - f^*g_2$ .  $M$  is a slice if  $f$  is a constant map. The hyperbolic angle  $\theta$  can be defined by (see [2], [15])

$$\cosh \theta = \frac{1}{\sqrt{\det(g_1 - f^*g_2)}},$$

where the determinant is defined with respect to the metric  $g_1$ . The angle  $\theta$  measures the deviation from a spacelike submanifold to a slice. If this angle is bounded the metrics  $g_1$  and  $g = g_1 - f^*g_2$  of  $\Sigma_1$  are equivalent. In this case,  $(\Sigma_1, g_1)$  is compact iff  $(M, g)$  is so. The following is the main theorem in this paper.

**Theorem 1.1.** *Let  $f$  be a smooth map from  $\Sigma_1$  to  $\Sigma_2$  such that  $F_0 : M \rightarrow \bar{M}$  is a compact spacelike graph of  $f$ . We assume  $(\Sigma_1, g_1)$  closed of dimension  $m \geq 2$ ,  $(\Sigma_2, g_2)$  complete of dimension  $n \geq 1$ ,  $\text{Ricci}_1(p) \geq 0$  and  $K_1(p) \geq K_2(q)$  for any  $p \in \Sigma_1$ ,  $q \in \Sigma_2$  and the curvature tensor  $R_2$  of  $\Sigma_2$  and all its covariant derivatives are bounded. Then:*

- (1) *The mean curvature flow (1.1) of the spacelike graph of  $f$  remains a spacelike graph of a map  $f_t : \Sigma_1 \rightarrow \Sigma_2$  and exists for all time.*
- (2) *If  $\Sigma_2$  is also compact there is a sequence  $t_n \rightarrow +\infty$  such that the flow converges at infinity to a spacelike graph of a totally geodesic map, and if  $\text{Ricci}_1(p) > 0$  at some point  $p \in \Sigma_1$ , the sequence converges to a slice.*
- (3) *If  $\text{Ricci}_1 > 0$  everywhere, all the flow converges to a unique slice.*

We observe that in (3) we do not need the compactness assumption of  $\Sigma_2$ . We also note that the condition  $\text{Ricci}_1 \geq 0$  and  $K_1 \geq K_2$  means that at a point  $p \in \Sigma_1$ , if  $K_1(P) < 0$  at some two-plane  $P$  of  $T_p\Sigma_1$ , then we have to require  $K_2 < 0$  everywhere. In case (3) the flow defines a homotopy  $f_t(\phi_t(p))$  from the initial map  $f_0 = f$  to the final constant map  $f_\infty$ , where  $\phi_t = \pi_1 \circ F_t$  is a smooth diffeomorphic endomorphism of  $\Sigma_1$ , that at  $t = 0$  gives the identity map. We shall prove that the deformation process is also valid without assuming  $K_1 \geq K_2$ .

**Theorem 1.2.** *Suppose  $(\Sigma_1, g_1)$  and  $(\Sigma_2, g_2)$  are two complete Riemannian manifolds of dimensions  $m \geq 2$  and  $n \geq 1$  respectively,  $\Sigma_1$  closed,  $K_1 > 0$  everywhere, or  $\text{Ricci}_1 > 0$  and  $K_2 \leq -c$  with  $c > 0$  constant, and the curvature tensor of  $\Sigma_2$  and all its covariant derivatives are bounded. Then there exists a constant  $\rho \geq 0$ , depending only on  $\min K_1$  and on  $\sup K_2^+$ , such that any smooth map  $f : \Sigma_1 \rightarrow \Sigma_2$  satisfying  $f^*g_2 < \rho g_1$  can be homotopically deformed into a constant map.*

The constant  $\rho \in (0, +\infty]$  can be taken equal to  $\min_{\Sigma_1} K_1 / \sup_{\Sigma_2} K_2^+$ , where  $K_2^+ = \sup\{0, K_2\}$ , in case  $K_1 > 0$ , and equal to  $+\infty$  in case  $\text{Ricci}_1 > 0$  and  $K_2 \leq -c$ . Recall that by the Cartan-Hadamard theorem, if  $K_2 \leq 0$ , the universal cover of  $\Sigma_2$  is diffeomorphic to a Euclidean space, and in particular  $\pi_m(\Sigma_2) = 0$  for all  $m > 0$ . If  $\Sigma_1$  is the  $m$ -sphere, and  $K_2 \leq 0$  then  $\rho = +\infty$  and the previous corollary gives a new proof of this classical result.

**Corollary 1.1.** *If  $\Sigma_1$  is closed with  $K_1 > 0$  everywhere,  $K_2 \leq 0$  and for all  $k \geq 0$ ,  $\nabla^k R_2$  is bounded, then any map  $f : \Sigma_1 \rightarrow \Sigma_2$  is homotopically trivial.*

We also may apply the previous theorem to obtain a reformulated version of the main result in [21]:

**Corollary 1.2.** ([21]) *If  $\Sigma_1$  and  $\Sigma_2$  are compact with constant sectional curvatures  $\tau_1$  and  $\tau_2$  satisfying  $\tau_1 \geq |\tau_2|$ ,  $\tau_1 + \tau_2 > 0$ , and if  $\det(g_1 + f^*g_2) < 2$ , then  $\Gamma_f$  can be deformed by a family of graphs to the one of a constant map.*

The condition  $\det(g_1 + f^*g_2) < 2$  implies  $\Gamma_f$  is a spacelike submanifold for the pseudo-Riemannian structure of  $\Sigma_1 \times \Sigma_2$ . The converse may not hold, so spacelike graph is a less restrictive condition. In [19], corollary 1.2 is generalized, under the same constant curvature conditions, to the case of area decreasing maps that is a slightly less restrictive condition than of a spacelike graph  $f^*g_2 < g_1$ . For such maps the eigenvalues  $\lambda_i^2$  of  $f^*g_2$  satisfy  $\lambda_i \lambda_j < 1$  for  $i \neq j$ . Thus,  $f^*g_2$  may have one and only one eigenvalue (counting with multiplicity) greater than or equal to one. If  $n \geq 2$  area decreasing maps are spacelike iff the largest eigenvalue  $\lambda_1$  is also smaller than one. In this case we also recover the main theorem of [19]. If  $\Sigma_2$  is one-dimensional, any map  $f$  satisfies such condition, and the result can be obtained from theorem 1.2, since  $K_2 = 0$  holds in this case. This is a particular case of  $K_2 \leq 0$  stated above.

We consider the  $\phi$ -energy functional acting on smooth maps  $f : (\Sigma_1, g_1) \rightarrow (\Sigma_2, g_2)$ ,

$$E_\phi(f) = \int_{\Sigma_1} \phi(\lambda_1^2, \dots, \lambda_m^2) d\mu_1,$$

where  $d\mu_1$  means here the volume element of  $(\Sigma_1, g_1)$ ,  $\phi$  is a symmetric nonnegative continuous function on the eigenvalues  $\lambda = \lambda_i^2$  of  $f^*g_2$  satisfying  $\phi(\lambda) = 0$  if and only if  $\lambda = 0$  and  $\phi(\lambda) \leq C\|\lambda\|^\tau$ , for some constants  $C, \tau > 0$ . When  $\phi(\lambda) = \lambda_1^2 + \dots + \lambda_m^2$ , we have the usual energy functional whose critical points are the harmonic maps. As a corollary of theorem 1.2 we obtain:

**Corollary 1.3.** *Under the same curvature conditions of  $(\Sigma_i, g_i)$  given in theorem 1.2, if  $f : (\Sigma_1, g_1) \rightarrow (\Sigma_2, g_2)$  minimizes the  $\phi$ -energy functional in its homotopy class, and if  $f^*g_2 \leq \rho g_1$ , then  $f$  is constant.*

The rest of this paper is organized as follows. In section 2, we derive the elementary formulae for the geometry of spacelike submanifolds in a pseudo-Riemannian manifold. Section 3 is devoted to spacelike submanifolds in pseudo-Riemannian product manifolds in our setting. Evolution equations of different geometric quantities are given in section 4. In section 5 we prove part of theorem 1.1(1), and in section 6 we obtain long-time existence using elliptic Schauder theory and prove the existence of a convergent sequence of the flow. The use of the Bernstein-type results obtained in [2, 15] leads to theorem 1.1(2). In section 7 we consider the particular case  $\text{Ricci}_1 > 0$  everywhere, and  $\Sigma_2$  not necessarily compact, and prove the convergence of all the flow. In this section we also prove theorem 1.2 and corollaries 1.2 and 1.3.

## 2 Geometry of spacelike submanifolds

Let  $\bar{M}$  be an  $(m+n)$ -dimensional pseudo-Riemannian manifold, and  $\bar{g}$  the non-degenerate metric on  $\bar{M}$ , which is of index  $n$ . Denote by  $\bar{\nabla}$  the connection on  $\bar{M}$ , and we convention that the curvature tensor  $\bar{R}$  is defined by  $\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]}Z$ , and  $\bar{R}(X, Y, Z, W) = \bar{g}(\bar{R}(Z, W)Y, X)$ , for any tangent vector fields  $X, Y, Z$  and  $W$  of  $\bar{M}$ . Suppose  $F : M \rightarrow \bar{M}$  is a  $m$ -dimensional spacelike submanifold immersed into  $\bar{M}$ , i.e. the induced metric of  $M$  is positive definite. For any tangent vector fields  $X, Y$  of  $M$  and  $V$  a time-like normal vector,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X V = \nabla_X^\perp V - A_V X,$$

where  $\nabla$  is the induced connection on  $M$ , and  $\nabla_X^\perp V = (\bar{\nabla}_X V)^\perp$  the normal connection in the normal bundle  $NM$ , and  $B$  and  $A$  are the second fundamental form and the Weingarten transformation, respectively,  $g(A_V(X), Y) = \bar{g}(V, B(X, Y))$ .

We choose orthonormal frame fields  $\{e_1, \dots, e_{m+n}\}$  of  $\bar{M}$ , such that when restricting to  $M$ ,  $\{e_1, \dots, e_m\}$  is a tangent frame field, and  $\{e_{m+1}, \dots, e_{m+n}\}$  is

a normal frame field. We make use of the indices range,  $1 \leq i, j, k, \dots, \leq m$ ,  $m+1 \leq \alpha, \beta, \dots, \leq m+n$ , and  $1 \leq a, b, c, \dots, \leq m+n$ . Let  $\theta^1, \dots, \theta^{m+n}$  be the dual frame fields of  $\{e_a\}$ . Then the structure equations of  $\bar{M}$  are given by

$$d\theta^a = -\sum_b \theta_b^a \wedge \theta^b, \quad d\theta_b^a = -\sum_c \theta_c^a \wedge \theta_b^c + \Phi_b^a,$$

where  $\Phi_b^a = \frac{1}{2} \sum_{c,d} \bar{R}_{bcd}^a \theta^c \wedge \theta^d$  are the curvature forms, and  $\theta_a^b$  the connection forms satisfying  $\sum_c \bar{g}_{ac} \theta_b^c + \bar{g}_{cb} \theta_a^c = d\bar{g}_{ab} = 0$ . Let  $\omega^a = F^* \theta^a$ ,  $\omega_a^b = F^* \theta_a^b$ . Then restricting to  $M$ , we have  $\omega^\alpha = 0$ , and  $\sum_i \omega_i^\alpha \wedge \omega^i = 0$ . By Cartan lemma,

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \quad (2.1)$$

where  $h_{ij}^\alpha$  are the components of the second fundamental form, that is  $B(e_i, e_j) = \sum_\alpha h_{ij}^\alpha e_\alpha$ . Since the normal vectors are time-like, the following relations hold

$$\bar{g}(B(e_i, e_j), e_\alpha) = \bar{g}(A_\alpha e_i, e_j) = -h_{ij}^\alpha.$$

The structure equations of  $M$  are then given by

$$\begin{aligned} d\omega^i &= -\sum_j \omega_j^i \wedge \omega^j, \\ d\omega_j^i &= -\sum_k \omega_k^i \wedge \omega_j^k + \Omega_j^i, \\ d\omega_\beta^\alpha &= -\sum_\gamma \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \Omega_\beta^\alpha, \end{aligned}$$

where  $\nabla_{e_j} e_i = \sum_k \omega_j^k(e_i) e_k$ ,  $\nabla_{e_j}^\perp e_\alpha = \sum_\beta \omega_\alpha^\beta(e_j) e_\beta$ , with  $\omega_\alpha^\beta + \omega_\beta^\alpha = \omega_i^k + \omega_k^i = 0$ , and  $\Omega_j^i = \frac{1}{2} \sum_{k,l} R_{jkl}^i \omega^k \wedge \omega^l$ ,  $\sum_i R_{jkl}^i e_i = R(e_k, e_l) e_j$  is the curvature form of  $M$ , and  $\Omega_\beta^\alpha = \frac{1}{2} \sum_{k,l} R_{\beta kl}^\alpha \omega^k \wedge \omega^l$ ,  $R^\perp(e_j, e_k) e_\beta = \sum_\alpha R_{\beta jk}^\alpha e_\alpha$  is the normal curvature form. Setting  $\bar{R}(e_c, e_d) e_b = \sum_a \bar{R}_{bcd}^a e_a$  we have the Gauss equation

$$R_{jkl}^i = \bar{R}_{jkl}^i - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

and the normal curvature of  $M$  is given by the Ricci equation

$$R_{\beta kl}^\alpha = \bar{R}_{\beta kl}^\alpha - \sum_i (h_{ki}^\alpha h_{li}^\beta - h_{li}^\alpha h_{ki}^\beta).$$

The tensor given by  $\nabla_Z B(X, Y) = \nabla_Z^\perp(B(X, Y)) - B(\nabla_Z X, Y) - B(X, \nabla_Z Y)$  is the covariant derivative of  $B$ . The components of  $\nabla_{e_k} B(e_i, e_j) = \sum_\alpha h_{ij,k}^\alpha e_\alpha$  satisfy

$$\sum_k h_{ij,k}^\alpha \omega^k = dh_{ij}^\alpha - \sum_k h_{kj}^\alpha \omega_i^k - \sum_k h_{ik}^\alpha \omega_j^k + \sum_\beta h_{ij}^\beta \omega_\beta^\alpha. \quad (2.2)$$

Consider  $\tilde{R} : \wedge^2 TM \rightarrow L(TM; NM)$  the restriction of  $\bar{R}$ , defined for  $X, Y, Z \in T_p M$  and  $U \in NM_p$ ,  $\tilde{g}(\tilde{R}(X, Y)Z, U) = \tilde{g}(\bar{R}(X, Y)Z, U)$ . Then the components of  $\tilde{R}$  are just  $\tilde{R}_{ijk}^\alpha = \bar{R}_{aijk}\tilde{g}^{a\alpha} = -\bar{R}_{\alpha ijk}$ . Differentiating both sides of (2.1) and applying the structure equations we have the Codazzi equation

$$h_{ij,k}^\alpha - h_{ik,j}^\alpha = - \left( (\bar{R}(e_j, e_k)e_i)^\perp \right)^\alpha = -\bar{R}_{ijk}^\alpha.$$

The mean curvature of  $F$  is denoted by  $H = \text{trace } B = \sum_\alpha H^\alpha e_\alpha$ ,  $H^\alpha = \sum_i h_{ii}^\alpha$ . The tensor defined by  $\nabla_{X,Y}^2 B(Z, W) = (\nabla_Y(\nabla_X B) - \nabla_{\nabla_Y X} B)(Z, W)$  is the second covariant derivative of  $B$ . The components of  $\nabla_{e_k, e_l}^2 B(e_i, e_j) = \sum_\alpha h_{ij,kl}^\alpha e_\alpha$ , satisfy

$$\sum_l h_{ij,kl}^\alpha \omega^l = dh_{ij,k}^\alpha - \sum_l h_{lj,k}^\alpha \omega_i^l - \sum_l h_{il,k}^\alpha \omega_j^l - \sum_l h_{ij,l}^\alpha \omega_k^l + \sum_\beta h_{ij,k}^\beta \omega_\beta^\alpha.$$

Differentiation of (2.2) and use of the structure equations we have

$$h_{ij,kl}^\alpha - h_{ik,jl}^\alpha = \sum_r h_{ir}^\alpha R_{rjk}^l + \sum_r h_{rj}^\alpha R_{rik}^l - \sum_\beta h_{ij}^\beta R_{\beta kl}^\alpha. \quad (2.3)$$

In order to compute the Laplacian of the second fundamental form, we have to relate the covariant derivatives  $(\bar{\nabla}_{e_s} \bar{R}(e_j, e_k)e_i)^\alpha$ , with  $\nabla_{e_s} \tilde{R}(e_j, e_k)e_i = \sum_\alpha \tilde{R}_{ijk,s}^\alpha e_\alpha$ , where  $\nabla \tilde{R}$  is the covariant derivative considering the connection of the normal bundle. We have

$$(\bar{\nabla}_l \bar{R})_{ijk}^\alpha = \tilde{R}_{ijk,l}^\alpha - \sum_\beta \bar{R}_{\beta jk}^\alpha h_{il}^\beta - \sum_\beta \bar{R}_{i\beta k}^\alpha h_{jl}^\beta - \sum_\beta \bar{R}_{ij\beta}^\alpha h_{kl}^\beta + \sum_r \bar{R}_{ijk}^r h_{rl}^\alpha.$$

Using Codazzi's equation (4 times), we obtain

$$h_{ij,ks}^\alpha = h_{ik,js}^\alpha - \tilde{R}_{ijk,s}^\alpha$$

and so, using this equation again and the commutation formula (2.3), we get

$$\begin{aligned} h_{ij,kk}^\alpha &= h_{ki,jk}^\alpha - \tilde{R}_{ijk,k}^\alpha \\ &= h_{ki,kj}^\alpha + \sum_r h_{kr}^\alpha R_{rik}^j + \sum_r h_{ri}^\alpha R_{kkj}^r - \sum_\beta h_{ki}^\beta R_{\beta jk}^\alpha - \tilde{R}_{ijk,k}^\alpha \\ &= h_{kk,ij}^\alpha - \tilde{R}_{kik,j}^\alpha + \sum_r h_{kr}^\alpha R_{rik}^j + \sum_r h_{ri}^\alpha R_{kkj}^r - \sum_\beta h_{ki}^\beta R_{\beta jk}^\alpha - \tilde{R}_{ijk,k}^\alpha. \end{aligned}$$

Thus,

$$\begin{aligned}
\sum_k h_{ij,kk}^\alpha &= \sum_k (h_{ik,j}^\alpha - \tilde{R}_{ijk}^\alpha)_{,k} \\
&= \sum_k (h_{ki,jk}^\alpha - \tilde{R}_{ijk,k}^\alpha) \\
&= \sum_k (h_{ki,kj}^\alpha + \sum_r h_{kr}^\alpha R_{ijk}^r + \sum_r h_{ri}^\alpha R_{kjk}^r - \sum_\beta h_{ki}^\beta R_{\beta jk}^\alpha - \tilde{R}_{ijk,k}^\alpha) \\
&= \sum_k (h_{kk,ij}^\alpha - \tilde{R}_{kik,j}^\alpha + \sum_r h_{kr}^\alpha R_{ijk}^r + \sum_r h_{ri}^\alpha R_{kjk}^r - \sum_\beta h_{ki}^\beta R_{\beta jk}^\alpha - \tilde{R}_{ijk,k}^\alpha).
\end{aligned}$$

The Laplacian of  $B$  is the symmetric  $NM$ -valued 2-tensor of  $M$ ,  $\Delta B = \text{trace} \nabla_{\cdot}^2 B$ , that is  $(\Delta B(e_i, e_j))^\alpha = \sum_k h_{ij,kk}^\alpha = \Delta h_{ij}^\alpha$ . Then we have

$$\begin{aligned}
(\Delta B(e_i, e_j))^\alpha &= \Delta h_{ij}^\alpha = \\
&= H_{,ij}^\alpha + \sum_k (-\tilde{R}_{kik,j}^\alpha - \tilde{R}_{ijk,k}^\alpha + \sum_r h_{kr}^\alpha R_{ijk}^r + \sum_r h_{ri}^\alpha R_{kjk}^r - \sum_\beta h_{ki}^\beta R_{\beta jk}^\alpha) \\
&= H_{,ij}^\alpha + \sum_k \left( -(\bar{\nabla}_j \bar{R})_{kik}^\alpha - \sum_\beta \bar{R}_{\beta ik}^\alpha h_{kj}^\beta - \sum_\beta \bar{R}_{k\beta k}^\alpha h_{ij}^\beta - \sum_\beta \bar{R}_{ki\beta}^\alpha h_{jk}^\beta + \sum_l \bar{R}_{kik}^l h_{lj}^\alpha \right. \\
&\quad \left. - (\bar{\nabla}_k \bar{R})_{ijk}^\alpha - \sum_\beta \bar{R}_{\beta jk}^\alpha h_{ik}^\beta - \sum_\beta \bar{R}_{i\beta k}^\alpha h_{jk}^\beta - \sum_\beta \bar{R}_{ij\beta}^\alpha h_{kk}^\beta + \sum_l \bar{R}_{ijk}^l h_{kl}^\alpha \right. \\
&\quad \left. + \sum_r h_{kr}^\alpha [\bar{R}_{ijk}^r - \sum_\beta (h_{rj}^\beta h_{ik}^\beta - h_{rk}^\beta h_{ij}^\beta)] + \sum_r h_{ri}^\alpha [\bar{R}_{kjk}^r - \sum_\beta (h_{rj}^\beta h_{kk}^\beta - h_{rk}^\beta h_{kj}^\beta)] \right. \\
&\quad \left. - \sum_\beta h_{ki}^\beta [\bar{R}_{\beta jk}^\alpha - \sum_l (h_{jl}^\alpha h_{kl}^\beta - h_{kl}^\alpha h_{jl}^\beta)] \right).
\end{aligned}$$

Using the first Jacobi identity  $\bar{R}_{ij\beta}^\alpha = -\bar{R}_{\beta ij}^\alpha - \bar{R}_{j\beta i}^\alpha$ , and that  $\sum_{ij} h_{ij}^\alpha \bar{R}_{\beta ij}^\alpha = 0$ , we have

$$\begin{aligned}
-\langle B, \Delta B \rangle &= \sum_{ij\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{ij\alpha} \left( h_{ij}^\alpha H_{,ij}^\alpha - h_{ij}^\alpha [\sum_k (\bar{\nabla}_j \bar{R})_{kik}^\alpha + \sum_k (\bar{\nabla}_k \bar{R})_{ijj}^\alpha] \right. \\
&\quad \left. + \sum_{k\beta} (4\bar{R}_{\beta ki}^\alpha h_{kj}^\beta h_{ij}^\alpha - \bar{R}_{k\beta k}^\alpha h_{ij}^\alpha h_{ij}^\beta) + \sum_\beta \bar{R}_{i\beta j}^\alpha H^\beta h_{ij}^\alpha \right. \\
&\quad \left. + 2\sum_{kl} (\bar{R}_{ijk}^l h_{ij}^\alpha h_{kl}^\alpha + \bar{R}_{kik}^l h_{lj}^\alpha h_{ij}^\alpha) - \sum_{k\beta} h_{ij}^\alpha h_{jk}^\alpha h_{ki}^\beta H^\beta \right) \\
&\quad \left. + \sum_{ij\alpha\beta} [\sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha)]^2 + \sum_{\alpha\beta} (\sum_{ij} h_{ij}^\alpha h_{ij}^\beta)^2.
\end{aligned}$$

We obtain a Simons' type identity

$$\begin{aligned}
\Delta \|B\|^2 &= 2\|\nabla B\|^2 + \sum_{ij\alpha} 2h_{ij}^\alpha H_{,ij}^\alpha - \sum_{ij\alpha} 2h_{ij}^\alpha [\sum_k (\bar{\nabla}_j \bar{R})_{kik}^\alpha + \sum_k (\bar{\nabla}_k \bar{R})_{ijj}^\alpha] \\
&\quad + \sum_{ij\alpha\beta} 2\{ \sum_k (4\bar{R}_{\beta ki}^\alpha h_{kj}^\beta h_{ij}^\alpha - \bar{R}_{k\beta k}^\alpha h_{ij}^\alpha h_{ij}^\beta) + \bar{R}_{i\beta j}^\alpha H^\beta h_{ij}^\alpha \} \\
&\quad + \sum_{ijkl\alpha} 4(\bar{R}_{ijk}^l h_{ij}^\alpha h_{kl}^\alpha + \bar{R}_{kik}^l h_{lj}^\alpha h_{ij}^\alpha) - \sum_{ijk\alpha\beta} 2h_{ij}^\alpha h_{jk}^\alpha h_{ki}^\beta H^\beta \\
&\quad + 2\sum_{ij\alpha\beta} (\sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha))^2 + 2\sum_{\alpha\beta} (\sum_{ij} h_{ij}^\alpha h_{ij}^\beta)^2. \tag{2.4}
\end{aligned}$$

Notice that we use  $\|\cdot\|$  to denote the absolute of the norm of a time-like vector in  $\bar{M}$ .



### 3 $\Delta \cosh \theta$

In this section we shall compute the covariant derivative of a pull-back of a parallel form in the ambient space by a spacelike immersion  $F : M^m \rightarrow \bar{M}$ . Let  $\Omega$  be a parallel  $m$ -form on  $\bar{M}$ . For the orthonormal frame fields  $\{e_i, e_\alpha\}$  in section 2,  $\Omega(e_1, \dots, e_m)$  is a function on  $M$ . As in [21, 15], we shall compute the Laplacian of  $\Omega(e_1, \dots, e_m) = \Omega_{1\dots m}$  in locally frame fields. First we have

$$\begin{aligned} (\nabla_k F^* \Omega)(e_1, \dots, e_m) &= \sum_i \Omega(e_1, \dots, (\bar{\nabla}_k e_i - \nabla_k e_i), \dots, e_m) \\ &= \sum_i \Omega(e_1, \dots, B(e_k, e_i), \dots, e_m). \end{aligned} \quad (3.1)$$

Differentiating (3.1) again gives

$$\begin{aligned} (\Delta F^* \Omega)(e_1, \dots, e_m) &= \sum_i \Omega(e_1, \dots, \sum_k (\nabla_{e_k} B)(e_k, e_i) + (\bar{\nabla}_{e_k} B(e_k, e_i))^\top, \dots, e_m) \\ &\quad + \sum_k \sum_{j < i} \Omega(e_1, \dots, B(e_k, e_j), \dots, B(e_k, e_i), \dots, e_m) \\ &\quad + \sum_k \sum_{j > i} \Omega(e_1, \dots, B(e_k, e_i), \dots, B(e_k, e_j), \dots, e_m), \end{aligned}$$

where  $\Delta F^* \Omega = \sum_k \nabla_k \nabla_k F^* \Omega - \nabla_{\nabla_{e_k} e_k} F^* \Omega$  is the rough Laplacian. Using the Codazzi's equation  $\sum_k \nabla_{e_k}^\perp B(e_k, e_i) = \nabla_{e_i}^\perp H + (\bar{R}(e_k, e_i)e_k)^\perp$  and that

$$\sum_{ik} g((\bar{\nabla}_{e_k} B(e_k, e_i))^\top, e_i) = \sum_{ik} -\bar{g}(B(e_k, e_i), B(e_k, e_i)) = \|B\|^2,$$

we get in components

$$(\Delta F^* \Omega)_{1\dots m} = \Omega_{1\dots m} \|B\|^2 + 2 \sum_{\alpha < \beta, i < j} \Omega_{\alpha\beta ij} \hat{R}_{\beta ij}^\alpha + \sum_{\alpha, i} \Omega_{\alpha i} H_{,i}^\alpha - \sum_{\alpha, i, k} \Omega_{\alpha i} \bar{R}_{kik}^\alpha, \quad (3.2)$$

where  $H_{,i}^\alpha = (\nabla_{e_i}^\perp H)^\alpha$ , and

$$\hat{R}_{\beta ij}^\alpha = h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha, \quad \Omega_{\alpha\beta ij} = \Omega(e_1, \dots, e_\alpha, \dots, e_\beta, \dots, e_m)$$

with  $e_\alpha, e_\beta$  occupying the  $i$ -th and the  $j$ -th positions. The same meaning is for  $\Omega_{\alpha i}$ .

In the following we assume  $\bar{M} = \Sigma_1 \times \Sigma_2$  is a product of two Riemannian manifolds  $(\Sigma_i, g_i)$  of dimension  $m$  and  $n$ , with pseudo-Riemannian metric  $\bar{g} = g_1 - g_2$ . If we denote by  $\pi_i$  the projection from  $T\bar{M}$  onto  $T\Sigma_i$ , then for any  $X, Y \in T\bar{M}$ ,

$$\bar{g}(X, Y) = g_1(\pi_1(X), \pi_1(Y)) - g_2(\pi_2(X), \pi_2(Y)). \quad (3.3)$$

Suppose  $M$  is a spacelike graph of a smooth map  $f : \Sigma_1 \rightarrow \Sigma_2$ . For each  $p \in \Sigma_1$  let  $\lambda_1^2 \geq \lambda_2^2 \geq \dots \geq \lambda_m^2 \geq 0$  be the eigenvalues of  $f^*g_2$ . The spacelike condition on  $M$  means  $\lambda_i^2 < 1$ . By the classic Weyl's perturbation theorem [22], ordering the eigenvalues in this way, each  $\lambda_i^2 : \Sigma_1 \rightarrow [0, 1)$  is a continuous locally Lipschitz function. For each  $p$  let  $s = s(p) \in \{1, \dots, m\}$  be the rank of  $f$  at  $p$ , that is,  $\lambda_s^2 > 0$  and  $\lambda_{s+1} = \dots = \lambda_m = 0$ . Then  $s \leq \min\{m, n\}$ .

We take a  $g_1$ -orthonormal basis  $\{a_i\}_{i=1, \dots, m}$  of  $T_p\Sigma_1$  of eigenvectors of  $f^*g_2$  with corresponding eigenvalues  $\lambda_i^2$ . Set  $a_{i+m} = df(a_i) / \|df(a_i)\|$  for  $i \leq s$ . This constitutes an orthonormal system in  $T_{f(p)}\Sigma_2$ , that we complete to give an orthonormal basis  $\{a_\alpha\}_{\alpha=m+1, \dots, m+n}$  for  $T_{f(p)}\Sigma_2$ . Moreover, changing signs if necessary, we can write  $df(a_i) = -\lambda_{i\alpha} a_\alpha$ , where  $\lambda_{i\alpha} = \delta_{\alpha, m+i} \lambda_i$  meaning  $= 0$  if  $i > s$ , or  $\alpha > m + s$ . Therefore

$$e_i = \frac{1}{\sqrt{1 - \sum_\beta \lambda_{i\beta}^2}} (a_i + \sum_\beta \lambda_{i\beta} a_\beta) \quad i = 1, \dots, m \quad (3.4)$$

$$e_\alpha = \frac{1}{\sqrt{1 - \sum_j \lambda_{j\alpha}^2}} (a_\alpha + \sum_j \lambda_{j\alpha} a_j) \quad \alpha = m+1, \dots, m+n \quad (3.5)$$

form an orthonormal basis for  $T_p M$  and for  $N_p M$  respectively, with  $e_i$  a direct one.

From now on we take  $\Omega$  to be the volume form of  $\Sigma_1$ , which is a parallel  $m$ -form on  $\bar{M}$ . If  $M$  is a embedded  $m$ -submanifold such that for any  $p \in M$ , and a basis  $E_i$  of  $T_p M$ , the quantity  $\Omega(\pi_1(E_1), \dots, \pi_1(E_m))$  is non-null then  $M$  is locally a graph, for the later implies  $\pi_1 \circ F : M \rightarrow \Sigma_1$  is a local diffeomorphism. This means  $F(p) = (\phi(p), f(\phi(p)))$  where  $\phi : M \rightarrow \Sigma_1$  is a local diffeomorphism, and  $F$  can be locally identified with the graph  $\tilde{F}(p) = (p, f(p))$  up to parameterization. The mean curvature of  $F$  does not depend on the parameterization, only on its image. We shall call graphs to all such parameterizations. Note that by Lemma 3.1 of [2], if  $F$  is a spacelike submanifold with  $M$  compact,  $\phi : M \rightarrow \Sigma_1$  is a covering map, and so it is surjective. Hence, this map  $f : \Sigma_1 \rightarrow \Sigma_2$ , when locally defined and  $\Sigma_1$  compact, it is unique and globally defined.

Assume  $M = \Gamma_f$ . If  $M$  is a spacelike graph, then taking the orthonormal frame  $e_i$  as in (3.4)

$$\Omega_{1 \dots m} = \Omega(\pi_1(e_1), \dots, \pi_1(e_m)) = *F^* \Omega = \frac{1}{\sqrt{\prod_{i=1}^m (1 - \lambda_i^2)}} = \frac{1}{\sqrt{\det(g_1 - f^*g_2)}},$$

where  $*$  is the star operator in  $M$ . In this case this quantity is  $\geq 1$  (assuming the correct orientation) and is  $\cosh \theta$ . We can also describe  $\cosh \theta$  as the ratio between the volume elements of  $(\Sigma_1, g_1)$  and of  $(\Sigma_1, g = g_1 - f^* g_2)$ . If  $M$  is compact, any other submanifold in a sufficiently small neighbourhood of  $M$  is also a spacelike graph. Now we compute

$$2 \sum_{\alpha < \beta, i < j} \Omega_{\alpha\beta ij} \hat{R}_{\beta ij}^\alpha = 2 \sum_{\alpha, \beta, k, i < j} \lambda_{i\alpha} \lambda_{j\beta} (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha) \cosh \theta. \quad (3.6)$$

As for the terms containing the curvatures of the ambient space, we denote by  $R_1$  and  $R_2$  the curvature tensor of  $\Sigma_1$  and  $\Sigma_2$ , respectively. We shall compute the curvatures  $\bar{R}$  of  $\bar{M}$  in terms of  $R_1$  and  $R_2$ . Now for the tangent frame field  $\{e_i\}$  (3.4) and normal frame field  $\{e_\alpha\}$  (3.5), since  $\bar{R}_{kik}^\alpha = \bar{R}_{\beta kik} \bar{g}^{\alpha\beta} = -\bar{R}_{\alpha kik}$ , we obtain

$$\begin{aligned} & -\bar{R}_{kik}^\alpha = \bar{R}(e_\alpha, e_k, e_i, e_k) \\ &= R_1(\pi_1(e_\alpha), \pi_1(e_k), \pi_1(e_i), \pi_1(e_k)) - R_2(\pi_2(e_\alpha), \pi_2(e_k), \pi_2(e_i), \pi_2(e_k)) \\ &= \frac{\sum_l \lambda_{l\alpha} R_1(a_l, a_k, a_i, a_k) - \sum_{\beta, \gamma, \delta} \lambda_{k\beta} \lambda_{i\gamma} \lambda_{k\delta} R_2(a_\alpha, a_\beta, a_\gamma, a_\delta)}{\sqrt{(1 - \sum_j \lambda_{j\alpha}^2)(1 - \lambda_i^2)(1 - \lambda_k^2)}}. \end{aligned}$$

Consider for  $i \neq j$  the two-planes  $P_{ij} = \text{span}\{a_i, a_j\}$ ,  $P'_{ij} = \text{span}\{a_{m+i}, a_{m+j}\}$ . Since  $\lambda_{i\alpha}$  is diagonal, we have

$$\begin{aligned} & - \sum_{\alpha, i, k} \Omega_{\alpha i} \bar{R}_{kik}^\alpha \\ &= \sum_{i, j} \frac{\cosh \theta \lambda_i^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} (R_1(a_i, a_j, a_i, a_j) - \lambda_j^2 R_2(a_{m+i}, a_{m+j}, a_{m+i}, a_{m+j})) \\ &= \cosh \theta \sum_{i, j \neq i} \left( \frac{\lambda_i^2}{(1 - \lambda_i^2)} K_1(P_{ij}) + \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \quad (3.7) \end{aligned}$$

Inserting (3.6) and (3.7) into (3.2) we at last arrive at

$$\begin{aligned} \Delta \cosh \theta &= \cosh \theta \left\{ \|B\|^2 + 2 \sum_{k, i < j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j} - 2 \sum_{k, i < j} \lambda_i \lambda_j h_{ik}^{m+j} h_{jk}^{m+i} \right. \\ &+ \sum_i \left( \frac{\lambda_i^2}{(1 - \lambda_i^2)} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \Big\} \\ &+ \sum_{\alpha, i} \Omega_{\alpha i} H_{,i}^\alpha, \quad (3.8) \end{aligned}$$

where we have used the fact that the Hodge star operator is parallel. Now by (3.1) we have  $d \cosh \theta(e_k) = \cosh \theta \sum_i \lambda_i h_{ik}^{m+i}$ , which implies

$$\frac{|\nabla \cosh \theta|^2}{\cosh^2 \theta} = \sum_k \left( \sum_i \lambda_i h_{ik}^{m+i} \right)^2 = \sum_{i,k} (\lambda_i h_{ik}^{m+i})^2 + 2 \sum_{i < j, k} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j}. \quad (3.9)$$

We shall calculate

$$\Delta \ln(\cosh \theta) = \frac{\cosh \theta \Delta(\cosh \theta) - |\nabla \cosh \theta|^2}{\cosh^2 \theta}. \quad (3.10)$$

Plugging (3.8) and (3.9) into (3.10) we have

$$\begin{aligned} \Delta \ln(\cosh \theta) &= \|B\|^2 - \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{k, i < j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j} \\ &\quad + \sum_i \left( \frac{\lambda_i^2}{(1 - \lambda_i^2)} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \\ &\quad + (\cosh \theta)^{-1} \sum_{\alpha, i} \Omega_{\alpha i} H_i^\alpha. \end{aligned} \quad (3.11)$$

## 4 Evolution equations

In this section, we shall compute the evolution equations of several geometric quantities along the mean curvature flow (1.1). We fix a point  $(x_0, t_0) \in M \times [0, T)$  and consider  $(x, t)$  in a neighbourhood of  $(x_0, t_0)$ . We locally identify  $M_t = (M, g_t = F_t^* \bar{g})$  with  $(F_t(M), \bar{g}_{F_t(M)})$ . We take  $e_\alpha(x, t)$  a local o.n. frame of  $NM_t$  defined for  $(x, t)$  near  $(x_0, t_0)$ . Computations are easier considering a fixed local coordinate chart on  $M$ . For any local coordinate  $\{x^i\}$  on  $M$ , we use the same notation as in section 3, but  $g_{ij}(x, t) = g_t(\partial_i, \partial_j) = \delta_{ij}$  may not hold everywhere, and  $h_{ij}^\alpha$  the components of the second fundamental form  $B(x, t)$  are with respect to  $\partial_i$  and some orthonormal frame  $e_\alpha(x, t)$ . That is

$$g_{ij}(x, t) = \bar{g}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right), \quad h_{ij}^\alpha(x, t) = -\bar{g}(B(\partial_i, \partial_j), e_\alpha).$$

Most of the following computations are quite well known in the literature (see for instance [4, 11, 24]), but for the sake of simplicity, and since we are in the non-flat pseudo-Riemannian setting and in higher codimension, we reproduce them here adapted to our case. We define the tensor on  $M$  (depending on  $t$ )

$$\mathcal{H}(X, Y) = -\bar{g}(B(X, Y), H)$$

and  $\mathcal{H}_{ij} = \mathcal{H}(\partial_i, \partial_j)$ . Using the symmetry of the Hessian of  $F : M \times [0, T) \rightarrow \bar{M}$  ( $M$  with the initial metric  $g_0$ ), and that  $\nabla_{\frac{d}{dt}} \partial_i = \nabla_{\partial_i} \frac{d}{dt} = 0$ , we have

$$\bar{\nabla}_H \frac{\partial F}{\partial x^i} = \nabla_{\frac{d}{dt}} dF(\partial_i) = \bar{\nabla}_{\partial_i} \frac{d}{dt} F = \bar{\nabla}_{\partial_i} H.$$

Then,  $\frac{d}{dt} g_{ij} = \bar{g}(\bar{\nabla}_{\partial_i} H, dF(\partial_j)) + \bar{g}(dF(\partial_i), \bar{\nabla}_{\partial_j} H)$ . It follows the induced metric evolves according to

$$\begin{aligned} \frac{d}{dt} g_{ij} &= \sum_{\alpha} 2H^{\alpha} h_{ij}^{\alpha} = 2\mathcal{H}_{ij} \\ \frac{d}{dt} g^{ij} &= -\sum_{kr\alpha} 2g^{ik} g^{rj} H^{\alpha} h_{kr}^{\alpha} = -\sum_{kr} 2g^{ik} g^{rj} \mathcal{H}_{kr} \end{aligned} \quad (4.1)$$

for  $(x, t)$  near  $(x_0, t_0)$ . The volume element of  $M_t$  is given by  $d\mu_t = \text{Vol}_{M_t} = \sqrt{\det[g_{ij}]} dx^{1\dots m}$ . To compute the evolution equation for  $d\mu_t$  and for the second fundamental form we will assume the coordinate chart  $x^i$  is normal at  $x_0$  for the metric  $g_{t_0}$  with  $\partial_i(x_0) = e_i(x_0)$  orthonormal frame. Then at  $(x_0, t_0)$ ,  $g_{ij} = \delta_{ij}$ . The next computations are at the point  $(x_0, t_0)$ . Using (4.1)

$$\frac{d}{dt} \Big|_{t=t_0} d\mu_t = \frac{1}{2} \sum_k \frac{dg_{kk}}{dt} d\mu_{t_0} = \|H\|^2 d\mu_{t_0}.$$

We also have

$$\begin{aligned} \frac{d}{dt} h_{ij}^{\alpha} &= -\frac{d}{dt} \bar{g}(\bar{\nabla}_{\partial_j} \partial_i, e_{\alpha}) \\ &= -\bar{g}(\bar{\nabla}_H \bar{\nabla}_{\partial_j} \partial_i, e_{\alpha}) - \bar{g}(\bar{\nabla}_{\partial_j} \partial_i, \bar{\nabla}_H e_{\alpha}) \\ &= -\bar{g}(\bar{\nabla}_{\partial_j} \bar{\nabla}_H \partial_i + \bar{R}(H, \partial_j) \partial_i, e_{\alpha}) - \bar{g}(\bar{\nabla}_{\partial_j} \partial_i, \bar{\nabla}_H e_{\alpha}) \\ &= -\bar{g}(\bar{\nabla}_{\partial_j} \bar{\nabla}_{\partial_i} H, e_{\alpha}) - \bar{g}(\bar{R}(H, \partial_j) \partial_i, e_{\alpha}) - \bar{g}(\bar{\nabla}_{\partial_j} \partial_i, \bar{\nabla}_H e_{\alpha}). \end{aligned}$$

Set  $H_{,ij}^{\alpha} = (\nabla_{e_i, e_j}^2 H)^{\alpha}$ . Since at  $(x_0, t_0)$ ,  $\nabla_{\partial_i} \partial_j = 0$  then

$$\begin{aligned} \bar{g}(\bar{\nabla}_{\partial_j} \bar{\nabla}_{\partial_i} H, e_{\alpha}) &= -H_{,ij}^{\alpha} - \sum_{k\beta} H^{\beta} h_{ik}^{\beta} h_{jk}^{\alpha} \\ \bar{g}(\bar{\nabla}_{\partial_j} \partial_i, \bar{\nabla}_H e_{\alpha}) &= \sum_{\beta} h_{ij}^{\beta} \bar{g}(e_{\beta}, \bar{\nabla}_H e_{\alpha}). \end{aligned}$$

Thus,

$$\frac{d}{dt} h_{ij}^{\alpha} = H_{,ij}^{\alpha} + \sum_{k\beta} H^{\beta} h_{ik}^{\beta} h_{jk}^{\alpha} - \sum_{\beta} H^{\beta} \bar{R}_{i\alpha j\beta} - h_{ij}^{\beta} \bar{g}(e_{\beta}, \bar{\nabla}_H e_{\alpha})$$

and using  $\sum_{\alpha\beta} h_{ij}^\alpha h_{ij}^\beta \bar{g}(e_\beta, \bar{\nabla}_H e_\alpha) = 0$ , we have at  $(x_0, t_0)$

$$\begin{aligned} \frac{d}{dt} \|B\|^2 &= \sum_{ijls} \frac{d}{dt} (g^{il} g^{js} h_{ij} h_{ls}) \\ &= \sum_{ijk\alpha\beta} -4H^\beta h_{ij}^\beta h_{kj}^\alpha h_{ki}^\alpha + \sum_{ij\alpha} 2h_{ij}^\alpha \frac{d}{dt} h_{ij}^\alpha \\ &= \sum_{ij\alpha} 2h_{ij}^\alpha H_{,ij}^\alpha - \sum_{ij\alpha\beta} (2H^\beta h_{ij}^\beta h_{kj}^\alpha h_{ki}^\alpha + 2H^\beta \bar{R}_{\alpha j\beta} h_{ij}^\alpha). \end{aligned}$$

Combining the above equation with the Simon's type identity (2.4) we arrive last to the evolution equation of the squared norm of the second fundamental form as stated in next proposition. A similar computation can be done to  $\|H\|^2$ . Therefore

**Proposition 4.1.** *Let  $F : M \times [0, T) \rightarrow \bar{M}$  be an  $m$ -dimensional mean curvature flow of a spacelike submanifold in a pseudo-Riemannian manifold  $\bar{M}$ . Then the following evolution equations hold at  $(x_0, t_0)$*

$$\begin{aligned} \frac{d}{dt} d\mu_t &= \|H\|^2 d\mu_t \\ \frac{d}{dt} \|H\|^2 &= \Delta \|H\|^2 - 2\|\nabla^\perp H\|^2 - 4\|\mathcal{H}\|^2 - 2\text{trace}_g \bar{R}(dF(\cdot), H, dF(\cdot), H) \\ \frac{d}{dt} \|B\|^2 &= \Delta \|B\|^2 - 2\|\nabla B\|^2 + \sum_{ij\alpha} 2h_{ij}^\alpha (\sum_k (\bar{\nabla}_j \bar{R})_{kik}^\alpha + (\bar{\nabla}_k \bar{R})_{ijk}^\alpha) \\ &\quad - 2 \left( \sum_{ijk\alpha\beta} (4\bar{R}_{\beta ki}^\alpha h_{kj}^\beta h_{ij}^\alpha - \bar{R}_{k\beta k}^\alpha h_{ij}^\alpha h_{ij}^\beta) + \sum_{ijkl\alpha} 2(\bar{R}_{ijk}^l h_{ij}^\alpha h_{kl}^\alpha + \bar{R}_{kik}^l h_{lj}^\alpha h_{ij}^\alpha) \right) \\ &\quad - 2\sum_{ij\alpha\beta} (\sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha))^2 - 2\sum_{\alpha,\beta} (\sum_{ij} h_{ij}^\alpha h_{ij}^\beta)^2. \end{aligned}$$

Next we compute the evolution of the pull-back of a parallel  $m$ -form on  $\bar{M}$ . Let  $\Omega$  be a parallel  $m$ -form on  $\bar{M}$ . Then the restriction of  $F^*\Omega$  satisfies the following evolution equation at  $(x_0, t_0)$

$$\begin{aligned} \frac{d}{dt} (F^*\Omega(\partial_1, \dots, \partial_m)) &= \frac{d}{dt} (\Omega(F_*\partial_1, \dots, F_*\partial_m)) \\ &= \sum_i \Omega(\partial_1, \dots, \nabla_{\partial_i}^\perp H, \dots, \partial_m) + \Omega(\partial_1, \dots, -A_H \partial_i, \dots, \partial_m) \\ &= \sum_{\alpha,i} \Omega(\partial_1, \dots, e_\alpha, \dots, \partial_m) H_{,i}^\alpha + \Omega(\partial_1, \dots, \partial_m) \sum_\alpha (H^\alpha)^2 \\ &= \sum_{\alpha,i} \Omega_{\alpha i} H_{,i}^\alpha + \cosh \theta \|H\|^2, \end{aligned}$$

On the other hand we have

$$\begin{aligned}
\frac{d}{dt}\Omega_{1\dots m} &= \frac{d}{dt}(\Omega(\partial_1, \dots, \partial_m) \frac{1}{\sqrt{g}}) \\
&= \frac{1}{\sqrt{g}} \frac{d}{dt} \Omega(\partial_1, \dots, \partial_m) - \frac{1}{\sqrt{g}} \|\mathbf{H}\|^2 \Omega(\partial_1, \dots, \partial_m) \\
&= \sum_{\alpha, i} \Omega_{\alpha i} H_{,i}^{\alpha}.
\end{aligned} \tag{4.2}$$

Combining with equation (3.2) we get the parabolic equation satisfied by  $\Omega_{1\dots m}$ :

**Proposition 4.2.** *Let  $M_t$  be an  $m$ -dimensional spacelike mean curvature flow in a pseudo-Riemannian manifold  $\bar{M}$  and  $\Omega$  a parallel  $m$ -form on  $\bar{M}$ . Then we have the following evolution equation at  $(x_0, t_0)$*

$$\begin{aligned}
\frac{d}{dt}\Omega_{1\dots m} &= \Delta\Omega_{1\dots m} - \Omega_{1\dots m} \|\mathbf{B}\|^2 \\
&\quad - 2 \sum_{\alpha < \beta, i < j} \Omega_{\alpha\beta ij} \hat{R}_{\beta ij}^{\alpha} + \sum_{\alpha, i, k} \Omega_{\alpha i} \bar{R}_{kik}^{\alpha}.
\end{aligned}$$

If  $M$  is a graph of  $f : \Sigma_1 \rightarrow \Sigma_2$ , and  $\Omega$  is the volume form of  $\Sigma_1$ , then, since  $\Sigma_1$  is compact, for sufficiently small  $t$ ,  $M_t$  is a spacelike graph, and so  $\cos \theta_t$  is defined and we have the evolution equation for  $\cosh \theta$  by inserting (3.11) into (4.2)

**Proposition 4.3.** *Let  $F_0 : M \rightarrow \bar{M}$  be an immersion such that  $M_0$  is a spacelike graph over  $\Sigma_1$ . If each  $M_t$  is a graph  $\Gamma_{f_t}$  of a map  $f_t : \Sigma_1 \rightarrow \Sigma_2$  along the mean curvature flow of  $F_0$  for  $t \in [0, T')$ ,  $T' \leq T$ , then  $\cosh \theta$  satisfies the following equation, using the frames (3.4) and (3.5)*

$$\begin{aligned}
\frac{d}{dt} \ln(\cosh \theta) &= \Delta \ln(\cosh \theta) + \\
&\quad - \left\{ \|\mathbf{B}\|^2 - \sum_{k, i} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{k, i < j} \lambda_i \lambda_j h_{ik}^{m+j} h_{jk}^{m+i} \right\} \\
&\quad - \sum_i \left( \frac{\lambda_i^2}{(1 - \lambda_i^2)} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right)
\end{aligned} \tag{4.3}$$

## 5 Short-time existence

In this section we give the proof of the first part of Theorem 1.1(1).

Let  $x^i$  be a coordinate chart of  $\Sigma_1$  on a neighbourhood of  $p_0 \in \Sigma_1$  and  $y^\alpha$  a coordinate chart of  $\Sigma_2$  on a neighbourhood of  $q_0 = f_0(p_0)$ . Note that  $x^i$  is identified with a coordinate chart in  $M_0$  as in section 4. In coordinates (1.1) means

$$\sum_{ij} g^{ij} \left( \frac{\partial F_t^a}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial F_t^a}{\partial x_k} \right) = -\bar{G}(x, t)^a + \frac{d}{dt} F_t^a, \quad a = 1, \dots, m+n \quad (5.1)$$

where  $\bar{G}(x, t)^a = \sum_{ijbc} g^{ij} (\bar{\Gamma}_{bc}^a \circ F_t) \frac{\partial F_t^b}{\partial x^i} \frac{\partial F_t^c}{\partial x^j}$ , and  $\Gamma_{ij}^k$  are the Christoffel symbols for the induced Riemannian metric  $g_t$  of  $M$  (that depends on the second derivatives of  $F_t$ , what makes the system to be not strictly parabolic) and  $\bar{\Gamma}_{bc}^a$  the ones of  $\bar{M}$ , in the coordinates charts  $x^i$  and  $w^a = (x^i, y^\alpha)$  respectively.

Since  $F = F_0$  is a spacelike graph of  $f = f_0 : \Sigma_1 \rightarrow \Sigma_2$  we recall that in [17], following [16], we have proved that for  $X, Y \in T_p \Sigma_1$ ,

$$\begin{aligned} B(X, Y) &= (\nabla_X^1 Y - \nabla_X Y, df(\nabla_X^1 Y - \nabla_X Y)) + (0, Hess f(X, Y)) \\ &= (0, Hess f(X, Y))^\perp, \\ H &= (Z, df(Z)) + (0, W) = (0, W)^\perp \end{aligned} \quad (5.2)$$

where  $Hess f$  is the Hessian of  $f$  with respect to the Levi-Civita connections  $\nabla^i$  of  $(\Sigma_i, g_i)$ ,  $W = trace_g Hess f$ , and  $Z$  is the vector field on  $\Sigma_1$  defined by  $g(Z, X) = g_2(W, df(X))$ . From the above expression of  $B$  we have observed in [15] that  $\Gamma_f$  is a totally geodesic submanifold of  $\bar{M}$  iff  $f : (\Sigma_1, g_1) \rightarrow (\Sigma_2, g_2)$  is a totally geodesic map. To see this, we have from (5.2) and using the frames (3.4) and (3.5), for  $u \in T_q(\Sigma_2)$ ,  $u = \sum_\alpha u^\alpha a_\alpha$ ,

$$(0, u)^\perp = \left( \sum_i \frac{\lambda_i}{1 - \lambda_i^2} u^{i+m} a_i, \sum_i \frac{1}{1 - \lambda_i^2} u^{i+m} a_{i+m} + \sum_{\alpha > 2m} u^\alpha a_\alpha \right).$$

Using these frames we also see that  $\pi_1 : TM_p \rightarrow T_p \Sigma_1$  and  $\pi_2 : NM_p \rightarrow T_{f(p)} \Sigma_2$  define isomorphisms. From (5.2),

$$\begin{aligned} \sum_i \frac{1}{(1 - \lambda_i^2)} \|Hess f(\partial_k, \partial_j)^{i+m}\|_2^2 + \sum_{\alpha > m} \|Hess f(\partial_k, \partial_j)^\alpha\|_2^2 &= \|B(\partial_k, \partial_j)\|_2^2 \\ \sum_k ((\Gamma^1)_{ij}^k - \Gamma_{ij}^k) \partial_k &= \pi_1(B(\partial_i, \partial_j)) = \sum_k \frac{\lambda_k}{(1 - \lambda_k^2)} Hess f(\partial_i, \partial_j)^{k+m} a_k. \end{aligned} \quad (5.3)$$

We also note that if  $\lambda_i^2 < 1 - \delta$  for all  $i$ , the Riemannian metric  $\hat{g}$  on  $T_{(p, f(p))} \bar{M}$  defined by declaring an orthonormal basis  $e_i, e_\alpha$  given by (3.4) (3.5), that is,  $\hat{g} = \bar{g}|_{T\Gamma_f} - \bar{g}|_{T\Gamma_f^\perp}$ , is equivalent to the Riemannian metric  $\bar{g}_+ = g_1 + g_2$  of  $\bar{M}$  with



$c(\delta)\bar{g}_+ \leq \hat{g} \leq c'(\delta)\bar{g}_+$  along  $\Gamma_f$ , where  $c(\delta), c'(\delta)$  are positive constants that only depend on  $\delta$ .

If  $F_t : M_0 = \Sigma_1 \rightarrow \bar{M}$  is a graph,  $F_t(p) = (\phi_t(p), f_t(\phi_t(p)))$ , where  $\phi_t : \Sigma_1 \rightarrow \Sigma_1$  is given by  $\phi_t(p) = \pi_1(F_t(p))$  and satisfies  $\phi_0 = Id$ , then (1.1) means  $\frac{d\phi_t}{dt} = Z_t$ ,  $\frac{df_t}{dt} = W_t$ ,  $\phi_0 = Id$ , and  $f_{t=0} = f_0$ . In particular,  $f_t$  satisfies the evolution equation

$$\begin{cases} \frac{df}{dt} = W_t = \text{trace}_{g_t} \text{Hess } f_t, \\ f_{t=0} = f_0, \end{cases}$$

where the Hessian is w.r.t the initial metric  $g_1$  of  $\Sigma_1$  and the trace with respect to the graph metric  $g_t = g_1 - f_t^* g_2$  of  $\Sigma_1$ . This system is strictly parabolic.

Now we assume  $F_t$  satisfies (1.1). We identify  $M_0 = F_0(M)$  with the graph  $\Gamma_{f_0} : \Sigma_1 \rightarrow \bar{M}$ . We also remark that using the trick of DeTurck (see page 17 of [25]), as in the case of hypersurfaces in a Euclidean space, by reparameterizing  $F$  as  $\hat{F}(p, t) = F(\rho_t(p), t)$  where  $\rho_t : \Sigma_1 \rightarrow \Sigma_1$  is a convenient (local) diffeomorphism, (1.1) is equivalent to a system of strictly parabolic equations. For existence of short time solutions one can follow the approach in [7, 14] of isometrically embedding  $\Sigma_i$  into Euclidean spaces  $\mathbb{R}^{N_i}$ , but considering the Riemannian structures on  $\bar{M}$  and  $\mathbb{R}^{N_1+N_2}$ , and linearizing the above parabolic system to prove existence of a local solution. As we will see in next section, the tensor fields involved, namely  $\bar{\nabla}^k B$  are bounded both for the pseudo-Riemannian and the Riemannian structure of  $\bar{M}$ . Since  $\Sigma_1$  is compact one has:

**Proposition 5.1.** *A unique smooth solution of (1.1) with initial condition  $F_0$  a spacelike graphic submanifold exists in a maximal time interval  $[0, T)$  for some  $T > 0$ .*

Let  $T' \leq T$  such that for all  $t < T'$ ,  $M_t$  is an entire spacelike graph  $\Gamma_{f_t}$ , and  $\cosh \theta$  is bounded from above, that is  $\cosh \theta = 1/\sqrt{\prod_{i=1}^m (1 - \lambda_i^2)} < \Lambda$  for a constant  $\Lambda > 1$ . This is equivalent to  $\lambda_i^2 \leq 1 - \delta$  for some  $\delta > 0$  and any  $1 \leq i \leq m$ . We set

$$\eta_t := \max_{M_t} \cosh \theta. \quad (5.4)$$

Now we prove part (1) of theorem 1.1

**Proposition 5.2.**  *$T' = T$ , that is  $\cosh \theta$  has a finite upper bound, the evolving submanifold  $M_t$  remains a spacelike graph of a map  $f_t : \Sigma_1 \rightarrow \Sigma_2$  whenever the*

flow (1.1) exists. In particular  $f_t^* g_2$  and  $\|df_t\|^2$  (norm with respect to the initial metric  $g_1$  of  $\Sigma_1$ ) are uniformly bounded and the Riemannian metrics  $g_t$  on  $\Sigma_1$  are uniformly equivalent. Moreover

$$\int_0^T \sup_{\Sigma_1} \|H_t\|^2 dt < c_0$$

for some constant  $c_0 > 0$ .

*Proof.* Let  $t < T'$ . Note that  $\lambda_i \lambda_j < 1 - \delta$  for any  $i$  and  $j$ , and  $\lambda_i = 0$  for  $i > \min(m, n)$ . For the second fundamental form, we have

$$\|B\|^2 \geq \sum_{i,k,j} (h_{ik}^{m+j})^2 = \sum_{i < j, k} [(h_{ik}^{m+j})^2 + (h_{jk}^{m+i})^2] + \sum_{i,k} (h_{ik}^{m+i})^2,$$

where we keep in mind that  $h_{ik}^{m+j} = 0$  when  $m + j > m + n$ . Therefore we can estimate the terms in the bracket of (4.3)

$$\begin{aligned} & \|B\|^2 - \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{m+j} h_{jk}^{m+i} \\ & \geq \delta \|B\|^2 + (1 - \delta) \left\{ \sum_{i < j, k} [(h_{ik}^{m+j})^2 + (h_{jk}^{m+i})^2] + \sum_{i,k} (h_{ik}^{m+i})^2 \right\} \\ & \quad - (1 - \delta) \sum_{i,k} (h_{ik}^{m+i})^2 - 2(1 - \delta) \sum_{k,i < j} |h_{ik}^{m+j}| |h_{jk}^{m+i}| \\ & \geq \delta \|B\|^2. \end{aligned} \tag{5.5}$$

On the other hand, since  $\text{Ricci}_1 \geq 0$  and  $K_1(p) \geq K_2(q)$ , (4.4) is nonpositive. Thus by Proposition 4.3,  $\ln(\cosh \theta)$  satisfies the differential inequality for all  $t < T'$

$$\frac{d}{dt} \ln(\cosh \theta) \leq \Delta \ln(\cosh \theta) - \delta \|B\|^2 \leq \Delta \ln(\cosh \theta).$$

According to the maximal principle for parabolic equations, we have for  $s > t$

$$\eta_s \leq \eta_t \leq \eta_0. \tag{5.6}$$

Assume  $T' < T$ . Then  $F_{T'}$  is defined, and from  $F_t^* \Omega_{1\dots m} \geq 1$ , for all  $t < T'$  we obtain the same for  $t = T'$ . Then  $F_{T'}$  is a graph of a map  $f_{T'}$ . From (5.6) we have for all  $T' > t \geq 0$ ,  $\lambda_i^2(t) < 1$  and

$$1 - \lambda_i^2(t) \geq \prod_i (1 - \lambda_i^2(t)) \geq \frac{1}{\eta_t^2} \geq \frac{1}{\eta_0^2},$$

and so the same also holds for  $t = T'$ , what proves that  $f_{T'}$  defines a spacelike graph  $F_{T'}$ . Thus, we may take  $T' = T$ . Therefore,  $F_t(p) = (\phi_t(p), f_t(\phi_t(p)))$  with  $\phi_t : \Sigma_1 \rightarrow \Sigma_1$  a covering map homotopic to the identity, and so of degree one, and necessarily orientation preserving local diffeomorphism, what implies  $\phi_t$  is also 1-1. Now it follows that  $g_t$  are all uniformly equivalent and  $d\mu_t$  is uniformly bounded in  $\Sigma_1 \times [0, T)$ . From proposition 4.1,  $d\mu_t = e^{\int_0^t \|H_s\|^2 ds} d\mu_0$ , and so  $\sup_{\Sigma_1} \int_0^T \|H_s\|^2 ds < c_0$ , for some constant  $c_0 > 0$ . If we take  $p_s \in \Sigma_1$  such that  $\|H_s\|(p_s) = \max_{\Sigma_1} \|H_s\|$ , then we have  $\int_0^T \|H_s\|^2(p_s) ds \leq \sup_{\Sigma_1} \int_0^T \|H_s\|^2 ds < c_0$ .  $\square$

We will need the following lemmas:

**Lemma 5.1.** [6] *Let  $f$  be a function on  $M \times [0, T_1]$  satisfying*

$$\left(\frac{d}{dt} - \Delta\right)f \leq -a^2 f^2 + b^2$$

*for some constants  $a, b \in \mathbb{R}$ . Then we have  $f \leq \frac{b}{a} + \frac{1}{a^2 t}$  everywhere on  $M \times (0, T_1]$ .*

**Lemma 5.2.** [9] *Let  $\Sigma_1$  be a compact Riemannian manifold and  $f \in C^1(\Sigma_1 \times J)$  where  $J$  is an open interval, then  $f_{\max}(t) = \max_{\Sigma_1} f(\cdot, t)$  is Lipschitz continuous and there holds a.e.  $\frac{df_{\max}}{dt}(t) = \frac{df}{dt}(x_t, t)$ , where  $x_t \in \Sigma_1$  is a point which the maximum is attained.*

The Riemannian metrics  $\hat{g} = \bar{g}|_{TM_t} - \bar{g}|_{TM_t^\perp}$  on  $\bar{M}$  defined along the flow are uniformly equivalent to  $\bar{g}_+$ . Therefore, if  $U$  is a vector field of  $\bar{M}$  defined along the flow, and if  $U$  is normal or tangential to the flow, that is  $U = U^\perp$  or  $U = U^\top$ , then  $U$  is uniformly  $\bar{g}$ -bounded if and only if it is uniformly  $\bar{g}_+$ -bounded. Hence, any vector field  $U$  with  $U^\top$  and  $U^\perp$  uniformly  $\bar{g}$ -bounded, is also uniformly  $\bar{g}_+$ -bounded.

**Proposition 5.3.**  *$\|B\|$ ,  $\|H\|$ ,  $\|\nabla^k B\|$ , and  $\|\nabla^k H\|$ , for all  $k$ , are uniformly bounded as long as the solution exists. Furthermore  $\|B\|_{\bar{g}_+}$ ,  $\|\bar{\nabla}^k B\|_{\bar{g}_+}$  and  $\|\bar{\nabla}^k H\|_{\bar{g}_+}$  are uniformly bounded as well.*

*Proof.* In our case  $\Sigma_i$  are of bounded curvature tensors and their covariant derivative. By proposition 5.2,  $e_i$  and  $e_\alpha$  given by (3.4) and (3.5) are uniformly bounded. Thus, the terms in the expressions in Proposition 4.1 involving the curvature tensor  $\bar{R}$  are bounded. It follows from Proposition 4.1, for some constants  $c_1, c_2, c_3 \geq 0$ ,

$$\frac{d}{dt}\|B\|^2 \leq \Delta\|B\|^2 + c_1\|B\| + c_2\|B\|^2 - \frac{2}{n}\|B\|^4 \leq \Delta\|B\|^2 - \frac{1}{n}\|B\|^4 + c_3, \quad (5.7)$$

where we have used some elementary geometric-arithmetic inequality

$$\sum_{\alpha, \beta} (\sum_{ij} h_{ij}^\alpha h_{ij}^\beta)^2 \geq \sum_{\alpha} (\sum_{ij} (h_{ij}^\alpha)^2)^2 \geq \frac{1}{n} (\sum_{i, j, \alpha} (h_{ij}^\alpha)^2)^2 = \frac{1}{n} \|B\|^4.$$

Then applying Lemma 5.1 to (5.7),  $\|B\|$  is uniformly bounded. Then we proceed as in [5, 6, 10, 11, 12, 13, 18] to prove boundedness of  $\|\nabla^k B\|$ , using an interpolation formula for tensors. We note that all terms including the curvatures of ambient space are of lower orders of the second fundamental form than the main part. Since  $\nabla^k H = \text{trace} \nabla^k B$  we obtain uniform boundedness for  $\nabla^k H$  for all  $k \geq 0$ .  $\nabla^k B$  is the  $k$ -order covariant derivative of  $B$  using the normal connection  $\nabla^\perp$  of  $NM$ . Now we prove boundedness of  $\bar{\nabla}^k B$  in  $TM$  for the Riemannian structure  $\bar{g}_+ = g_1 + g_2$ . For  $k = 0$ , by the first equation of (5.3), uniform boundedness of  $\|B\|$  and of  $\lambda_i$  implies uniform boundedness of  $\|Hess f\|_2$ . It follows now by second equation of (5.3) we get uniform boundedness of  $\|\pi_1(B)\|_1$  and of  $\Gamma_{ij}^k$ . In particular  $\|\pi_2(B)\|_2$  is also bounded, and this proves uniform boundedness of  $\|B\|_{\bar{g}_+}$ . For  $k = 1$ , we have  $\bar{\nabla}_{\partial_s} B(\partial_i, \partial_j) = (\bar{\nabla}_{\partial_s} B(\partial_i, \partial_j))^\top + \nabla_{\partial_s} B(\partial_i, \partial_j)$  and so

$$\hat{g}(\bar{\nabla}_{\partial_s} B(\partial_i, \partial_j), \bar{\nabla}_{\partial_s} B(\partial_i, \partial_j)) = \sum_{\alpha \beta l r} g^{lr} h_{ij}^\alpha h_{sl}^\alpha h_{ij}^\beta h_{sr}^\beta + \|\nabla_{\partial_s} B(\partial_i, \partial_j)\|^2$$

that is,  $\|\bar{\nabla} B\|_{\hat{g}}^2 = \|(\bar{\nabla} B)^\top\|^2 + \|\nabla B\|^2 \leq c_{22} \|B\|^4 + \|\nabla B\|^2$ , with  $c_{22} > 0$  a constant not depending on  $t$ . Thus,  $\bar{\nabla} B$  is uniformly  $\hat{g}$ -bounded and so  $\bar{g}_+$  uniformly bounded. Inductively we obtain the same for higher order derivatives.  $\square$

**Corollary 5.1.**  $f^* g_2$  and  $\Gamma_{ij}^r$  and their derivatives are uniformly bounded.

*Proof.* Uniform boundedness of  $\|\bar{\nabla}^r B\|_{\bar{g}_+}$  implies by (5.2) and proposition 5.2, inductively on  $r \geq 0$ , uniform boundedness of  $\nabla_{\partial_s}^r \Gamma_{ij}^k$  and of  $\|\nabla^r Hess f\|$ . Since

$$\nabla_{\partial_s} f^* g_2(\partial_i, \partial_j) = g_2(Hess f(\partial_s, \partial_i), df(\partial_j)) + g_2(df(\partial_i), Hess f(\partial_s, \partial_j))$$

we obtain the uniform boundedness of  $f^* g_2$  and its derivatives.  $\square$

## 6 Long-time existence and convergence

It is well known that if  $\|B\|$  is uniformly bounded then the mean curvature flow exists for all time. This is well known for hypersurfaces, as in the above references,

and for the case of flat ambient space. For non flat space and higher codimension in the Riemannian case see [3, 4, 18]). This holds as well in our setting. For the sake of completeness we will apply Schauder theory for elliptic systems to prove long-time existence and a condition for the convergence of the flow at infinity.

We are assuming  $(\Sigma_1, g_1)$  compact and  $(\Sigma_2, g_2)$  complete. In this section we consider  $\bar{M}$  with the Riemannian metric  $\bar{g}_+ = g_1 + g_2$  and we may embed isometrically  $\Sigma_i$  into an Euclidean space  $\mathbb{R}^{N_i}$ , and consider  $\mathbb{R}^N$ ,  $N = N_1 + N_2$  with its Euclidean structure. Note that  $\bar{M}$  is a closed subset of  $\mathbb{R}^N$ , and so if  $K \subset \mathbb{R}^N$  is a compact set for the Euclidean topology, then  $K \cap \bar{M}$  is a compact set for  $(\bar{M}, \bar{g}_+)$ . Then we follow as in [14]. For each  $0 \leq \sigma < 1$  and  $k < +\infty$  integer, the spaces  $C^{k+\sigma}(\Sigma_1, \bar{M})$  are endowed with the usual  $C^{k+\sigma}$ -Hölder norms (well defined up to equivalence using coordinates charts).  $C^{k+\sigma}(\Sigma_1, \bar{M})$  is a Banach manifold with tangent space  $C^{k+\sigma}(\Sigma_1, F^{-1}T\bar{M})$  at  $F \in C^{k+\sigma}(\Sigma_1, \bar{M})$ . These spaces can be seen as closed subsets of  $C^{k+\sigma}(\Sigma_1, \mathbb{R}^N)$ . We consider  $F_t$  a solution of (1.1) with initial condition  $F_0 = \Gamma_{f_0}$ . Then  $F_t$  satisfies the parabolic system 5.1. Set  $a_{ij} = g^{ij}(x)$ , and  $b_k = \sum_{ij} g^{ij} \Gamma_{ij}^k(x)$ . From the computations in section 4, propositions 5.2 and 5.3, and corollary 5.1, we have that  $a_{ij}$ ,  $b_i$ ,  $\frac{d}{dx_k} a_{ij}$ ,  $\frac{d}{dx_k} b_{ij}$  (as well  $\frac{d}{dt} a_{ij}$  and  $\frac{d}{dt} b_{ij}$  as one can easily check) are uniformly bounded in  $\Sigma_1$ , and so  $a_{ij}$  and  $b_k$  are uniformly  $C^1$ -bounded in  $\Sigma_1$ . Note that if a vector field  $V$  of  $\bar{M}$  along the flow, is  $C^1(\Sigma_1, F^{-1}T\bar{M})$  -uniformly bounded for  $\bar{g}_+$ , then it is also in  $\mathbb{R}^N$ . The same conclusion for the higher order Hölder norms. Note also from proposition 5.2,  $\|df_t\|^2$  is uniformly bounded, and so  $\|\frac{\partial F_t^a}{\partial x_i}\|_{\bar{g}_+}$  is uniformly bounded. From uniform boundedness of  $\|B\|$  and of  $\Gamma_{ij}^k$  established in corollary 5.1, and that  $\frac{\partial^2 F^a}{\partial x_i \partial x_j} = B(\partial_i, \partial_j)^a + \sum_k \Gamma_{ij}^k(x) \frac{\partial F^a}{\partial x_k}$ , we have uniform boundedness of  $\|\frac{\partial F_t^a}{\partial x_i}\|_{C^\sigma(\Sigma_1, \mathbb{R}^N)}$ . By elliptic Schauder theory (see [14] p. 79), we have that a solution  $F_t$  of (5.1) satisfies for each  $t$

$$\begin{aligned} \|F(\cdot, t)\|_{C^{1+\sigma}(\Sigma_1, \bar{M})} &\leq c_4 (\|\bar{G}(\cdot, t)\|_{L^\infty(\Sigma_1, \mathbb{R}^N)} + \|H\|_{L^\infty(\Sigma_1, \mathbb{R}^N)}) \\ \|F(\cdot, t)\|_{C^{2+\sigma}(\Sigma_1, \bar{M})} &\leq c_5 (\|\bar{G}(\cdot, t)\|_{C^\sigma(\Sigma_1, \mathbb{R}^N)} + \|H\|_{C^\sigma(\Sigma_1, \mathbb{R}^N)}), \end{aligned}$$

Here  $c_i, i = 0, 1, \dots$  are positive constants not depending on  $t$ . Since  $\|H\|_{C^{k+\sigma}(\Sigma_1, \mathbb{R}^N)}$  is uniformly bounded, we have

**Proposition 6.1.** *Let  $F_t$  be a solution of (1.1) for  $t \in [0, T)$ . If  $\|F(\cdot, t)\|_{L^\infty(\Sigma_1, \bar{M})}$  is uniformly bounded then  $\|F(\cdot, t)\|_{C^{2+\sigma}(\Sigma_1, \bar{M})}$  is uniformly bounded for  $t \in [0, T)$ . Using the uniform boundedness of  $\|\bar{\nabla}^k B\|_{\bar{g}_+}$  we conclude uniform boundedness of  $\|F(\cdot, t)\|_{C^{k+1+\sigma}(\Sigma_1, \bar{M})}$ .*

**Corollary 6.1.**  *$T = +\infty$  and there exists a sequence  $t_n \rightarrow +\infty$  such that  $\sup_{\Sigma_1} \|H_{t_n}\| \rightarrow 0$ , when  $n \rightarrow +\infty$ . Moreover, if  $\sup_{p \in \Sigma_1} d_2(f_t(p), f_0(p)) < c_8$  uniformly for  $t \in [0, +\infty)$ , where  $c_8 > 0$  is a constant, then  $f_{t_n} \rightarrow f_\infty$  when  $n \rightarrow +\infty$ , where  $f_\infty : \Sigma_1 \rightarrow \Sigma_2$  is a smooth totally geodesic map defining a graphic spacelike totally geodesic submanifold. Furthermore, if  $\text{Ricci}_1(p) > 0$  at some point  $p \in \Sigma_1$ , then  $f_\infty$  is constant.*

*Proof.* Note that  $F_t(p)$  is a curve in  $\bar{M}$  with derivative  $H_t$ . Let  $\bar{d}$  be the distance function on  $(\bar{M}, \bar{g}_+)$ . We have a uniform bound  $\|H_t\|_{g_+} \leq c_9$ , for a constant  $c_9 > 0$ , and from (1.1), for  $t \geq s$

$$F_t(p) = F_s(p) + \int_s^t H_\tau(p) d\tau \quad \text{and} \quad \bar{d}(F_t(p), F_s(p)) \leq \int_s^t \|H_\tau\|_{g_+} d\tau. \quad (6.1)$$

Assuming  $T < +\infty$  we have  $\|F_t\|_{L^\infty(\Sigma_1, \mathbb{R}^N)}$  uniformly bound for  $t \in [0, T)$ . Therefore, by proposition 6.1,  $\|F(\cdot, t)\|_{C^{2+\sigma}(\Sigma_1, \bar{M})}$  are uniformly bounded. We take a sequence  $t_N \rightarrow T$ . By the Ascoli-Arzelà theorem we may extract a subsequence  $t_n \rightarrow T$  of  $t_N$ , such that  $F(\cdot, t_n)$  converges uniformly to a map  $F(\cdot, T)$  in  $C^2(\Sigma_1, \bar{M})$ , with  $t_n \rightarrow T$ . This also implies  $F(\cdot, t)$  converges uniformly to  $F(\cdot, T)$  when  $t \rightarrow T$ . To see this we only have to note that for  $d(F_t, F_T) = \sup_{p \in \Sigma_1} \bar{d}(F_t(p), F_T(p))$  or  $d$  defined from the 2-Hölder norm, the following inequality holds:

$$d(F_t, F_T) \leq d(F_t, F_{t_n}) + d(F_{t_n}, F_T) \leq c_{10}|t - t_n| + d(F_{t_n}, F_T), \quad (6.2)$$

where  $c_{10} > 0$  is a constant, and in the last inequality we used proposition 5.3. We note that  $F_T$  is smooth, by using, by induction, higher order Schauder theory to sequential subsequences of  $F_{t_n}$ , and finally a diagonal one. Following the same reasoning as in the proof of proposition 5.2,  $F_T$  is a spacelike graph of a map  $f_T \in C^\infty(\Sigma_1, \Sigma_2)$ , and consequently we can extend the solution  $F_t$  to  $[0, T + \varepsilon)$  for some  $\varepsilon > 0$ , what is impossible. This proves  $T = +\infty$ . It follows from proposition 5.2 that

$$\int_0^{+\infty} \sup_{\Sigma_1} \|H_t\|^2 \leq c_{12},$$

for some constant  $c_{12} > 0$ . Consequently there exist  $t_N \rightarrow +\infty$  with  $\sup_{\Sigma_1} \|H_{t_N}\|^2 \rightarrow 0$ . Assuming  $f_t$  lies in a compact set of  $\Sigma_2$  we are assuming  $\|F_t\|_{L^\infty(\Sigma_1, \bar{M})} \leq C$  uniformly for  $t \in [0, +\infty)$ , what implies, as above in this proof, for  $t_n$  subsequence of  $t_N$ ,  $F_{t_n}$  converges to a map  $F_\infty \in C^\infty(\Sigma_1, \bar{M})$  when  $n \rightarrow +\infty$ , with  $F_\infty$  a spacelike graph of a map  $f_\infty \in C^\infty(\Sigma_1, \Sigma_2)$ . Now  $F_\infty$  must satisfy  $\sup_{\Sigma_1} \|H_\infty\|^2 = 0$ , that is  $\Gamma_{f_\infty}$  is maximal. From the Bernstein results in [2, 15] we conclude  $\Gamma_{f_\infty}$  is a totally

geodesic submanifold, or equivalently,  $f : (\Sigma_1, g_1) \rightarrow (\Sigma_2, g_2)$  is a totally geodesic map, and if  $\text{Ricci}_1 > 0$  somewhere, then  $f_\infty$  is a constant map.  $\square$

So we have proved parts (1) and (2) of theorem 1.1.

## 7 The case $\text{Ricci}_1 > 0$ everywhere

Next we assume  $\text{Ricci}_1 > 0$  everywhere, and prove the last part (3) of theorem 1.1, giving a particular version of the proof of theorem 1.1, with no need of using the Bernstein theorems obtained in [2, 15], but with a direct proof of convergence at infinity of all flow to a graph of a constant map.

**Lemma 7.1.** *If  $\text{Ricci}_1 > 0$  everywhere, then for  $\eta$  given in (5.4),*

$$1 \leq \eta_t^2 \leq 1 + c_{16}e^{-2c_{15}t}, \quad (7.1)$$

$$\lambda_i^2(p, t) \leq \frac{c_{16}e^{-2c_{15}t}}{(1 + c_{16}e^{-2c_{15}t})} \quad \forall i. \quad (7.2)$$

for some constants  $c_{15}, c_{16} > 0$ . Thus  $\eta_t \rightarrow 1$  when  $t \rightarrow +\infty$ .

*Proof.* The assumption on the sectional curvatures of  $\Sigma_1$  and  $\Sigma_2$  in Theorem 1.1 with the further assumption  $\text{Ricci}_1 > 0$  everywhere guarantees that for each  $i$  fixed,

$$\left( \frac{1}{(1 - \lambda_i^2)} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \geq c_{14}$$

for some constant  $c_{14} > 0$ . Then we have by Proposition 4.3 and the proof of the first part of Theorem 1.1 (see (5.5))

$$\frac{d}{dt} \ln(\cosh \theta) \leq \Delta \ln(\cosh \theta) - c_{15} \sum_i \lambda_i^2,$$

for a constant  $c_{15} > 0$ . Now recall that  $\lambda_i \leq 1$  and

$$1 \geq \prod_i (1 - \lambda_i^2) = 1 - \sum_i \lambda_i^2 + \sum_{i < j} \lambda_i^2 \lambda_j^2 - \sum_{i < j < k} \lambda_i^2 \lambda_j^2 \lambda_k^2 + \dots.$$

By induction on  $m$  we see that

$$A_m := \prod_{1 \leq i \leq m} (1 - \lambda_i^2) - 1 + \sum_{1 \leq i \leq m} \lambda_i^2 = A_{m-1} + \lambda_m^2 \left( 1 - \prod_{1 \leq i \leq m-1} (1 - \lambda_i^2) \right) \geq 0.$$

We therefore have,  $\cosh^{-2} \theta = \prod_i (1 - \lambda_i^2) \geq 1 - \sum_i \lambda_i^2$ . It follows that for the positive constant  $c_{15}$

$$\frac{d}{dt} \ln(\cosh \theta) \leq \Delta \ln(\cosh \theta) + c_{15} \left( \frac{1}{\cosh^2 \theta} - 1 \right). \quad (7.3)$$

If there exists a time  $t_0$  such that  $\eta = 1$ , then  $M_{t_0}$  is already a stable solution. So without loss of generality, we may assume  $\eta \neq 1$ . In view of (7.3) and lemma 5.2 we have

$$\frac{d\eta}{dt} \leq c_{15} \left( \frac{1 - \eta^2}{\eta} \right),$$

and so  $1 \leq \eta^2 \leq 1 + c_{16} e^{-2c_{15}t}$  where  $c_{16} = \sup_{\Sigma_1} \cosh^2 \theta_0 - 1$ . Consequently,

$$(1 - \lambda_i^2) \geq \prod_i (1 - \lambda_i^2) \geq \frac{1}{(1 + c_{16} e^{-2c_{15}t})}.$$

□

**Proposition 7.1.** *If  $\text{Ricci}_1 > 0$  everywhere, then  $f_t$  lies in a compact region of  $\Sigma_1$  and  $f_t$   $C^\infty$ -converges to a unique constant map when  $t \rightarrow +\infty$ .*

*Proof.* Set  $\varepsilon(t) = c_{16} e^{-2c_{15}t}$  and  $\delta(t) = 1 - \varepsilon(t)$ . By (7.2),  $\lambda_i^2 \leq \varepsilon(t)$ . On the other hand, from (5.5)

$$||B||^2 - \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{m+j} h_{jk}^{m+i} \geq \delta(t) ||B||^2.$$

It follows from Proposition 4.3 that

$$\frac{d}{dt} \cosh \theta \leq \Delta \cosh \theta - \delta(t) \cosh \theta ||B||^2.$$

Set  $\bar{\eta} = \cosh \theta$ . We have by (3.9)  $||\nabla \bar{\eta}||^2 \leq \varepsilon(t) \bar{\eta}^2 ||B||^2$ . Let  $p(t)$  be a positive function not less than 1 of  $t$ , and  $\phi = \bar{\eta}^p ||B||^2$ . We have

$$\begin{aligned} \frac{d}{dt} \bar{\eta}^p &= \bar{\eta}^p \left( \frac{dp}{dt} \ln \bar{\eta} + p \frac{1}{\bar{\eta}} \frac{d}{dt} \bar{\eta} \right) \\ &= \bar{\eta}^p \frac{dp}{dt} \ln \bar{\eta} + p \bar{\eta}^{p-1} \frac{d}{dt} \bar{\eta} \\ &\leq \bar{\eta}^p \frac{dp}{dt} \ln \bar{\eta} + p \bar{\eta}^{p-1} (\Delta \bar{\eta} - \delta \bar{\eta} ||B||^2) \\ &= \bar{\eta}^p \frac{dp}{dt} \ln \bar{\eta} + \Delta \bar{\eta}^p - p(p-1) \bar{\eta}^{p-2} |\nabla \bar{\eta}|^2 - p \delta \bar{\eta}^p ||B||^2 \\ &\leq \bar{\eta}^p \frac{dp}{dt} \ln \bar{\eta} + \Delta \bar{\eta}^p - p \delta \bar{\eta}^p ||B||^2, \end{aligned}$$



and using (5.7)

$$\begin{aligned}
\frac{d}{dt}\phi &\leq (\bar{\eta}^p \frac{dp}{dt} \ln \bar{\eta} + \Delta \bar{\eta}^p - p\delta \bar{\eta}^p ||B||^2) ||B||^2 \\
&\quad + \bar{\eta}^p (\Delta ||B||^2 + c_1 ||B|| + c_2 ||B||^2 - \frac{2}{n} ||B||^4) \\
&= \Delta \phi - 2\bar{\eta}^{-p} \nabla \bar{\eta}^p \nabla \phi + 2\bar{\eta}^{-p} |\nabla \bar{\eta}^p|^2 ||B||^2 \\
&\quad + \bar{\eta}^p \left[ c_1 ||B|| + \left( \frac{dp}{dt} \ln \bar{\eta} + c_2 \right) ||B||^2 - \left( p\delta + \frac{2}{n} \right) ||B||^4 \right].
\end{aligned}$$

Now  $2\bar{\eta}^{-p} |\nabla \bar{\eta}^p|^2 ||B||^2 \leq 2p^2 \varepsilon \bar{\eta}^p ||B||^4$ , and we have

$$\begin{aligned}
\frac{d}{dt}\phi &\leq \Delta \phi - 2\bar{\eta}^{-p} \nabla \bar{\eta}^p \nabla \phi \\
&\quad + \bar{\eta}^p \left[ c_1 ||B|| + \left( \frac{dp}{dt} \ln \bar{\eta} + c_2 \right) ||B||^2 + \left( 2p^2 \varepsilon - p\delta - \frac{2}{n} \right) ||B||^4 \right] \\
&= \Delta \phi - 2\bar{\eta}^{-p} \nabla \bar{\eta}^p \nabla \phi + \bar{\eta}^{-p} \left( 2p^2 \varepsilon - p\delta - \frac{2}{n} \right) \phi^2 \\
&\quad + c_1 \bar{\eta}^{\frac{p}{2}} \phi^{\frac{1}{2}} + \left( \frac{dp}{dt} \ln \bar{\eta} + c_2 \right) \phi.
\end{aligned}$$

Setting  $\psi = e^{\frac{1}{2}c_{15}t} \phi$ , then  $\psi$  satisfies the evolution inequality

$$\begin{aligned}
\frac{d}{dt}\psi &\leq \Delta \psi - 2\bar{\eta}^{-p} \nabla \bar{\eta}^p \nabla \psi + e^{-\frac{1}{2}c_{15}t} \bar{\eta}^{-p} \left( 2p^2 \varepsilon - p\delta - \frac{2}{n} \right) \psi^2 \\
&\quad + c_1 e^{\frac{1}{4}c_{15}t} \bar{\eta}^{\frac{p}{2}} \psi^{\frac{1}{2}} + \left( \frac{dp}{dt} \ln \bar{\eta} + c_2 + \frac{1}{2}c_{15} \right) \psi.
\end{aligned}$$

Taking  $p^2 = \frac{1}{n\varepsilon}$ , that is,

$$p(t) = \frac{e^{c_{15}t}}{\sqrt{nc_6}},$$

and using that  $\ln(1+a) \leq a$  for all  $a \geq 0$ , we obtain  $\ln \bar{\eta} \leq \frac{c_{16}}{2} e^{-2c_{15}t}$ . It is easy to check that

$$(1 + c_{16}e^{-2c_{15}t})^{\frac{p}{4}} = (1 + c_{16}e^{-2c_{15}t})^{\frac{e^{c_{15}t}}{4\sqrt{nc_{16}}}} \rightarrow 1, \text{ as } t \rightarrow \infty,$$

what implies  $1 \geq \bar{\eta}^{-p} \geq c_{13}$ , for some constant  $c_{13} > 0$ . Therefore

$$\begin{aligned}
\frac{d}{dt}\psi &\leq \Delta \psi - 2\bar{\eta}^{-p} \nabla \bar{\eta}^p \nabla \psi - \frac{c_{13}\delta}{\sqrt{nc_{16}}} e^{\frac{1}{2}c_{15}t} \psi^2 \\
&\quad + c_1 e^{\frac{1}{4}c_{15}t} (1 + c_{16}e^{-2c_{15}t})^{\frac{p}{4}} \psi^{\frac{1}{2}} + \left( \frac{c_{15}\sqrt{c_{16}}}{2\sqrt{n}} e^{-c_{15}t} + c_2 + \frac{1}{2}c_{15} \right) \psi.
\end{aligned}$$

and by choosing  $T_0$  large enough such that for  $t \geq T_0$  one has  $\delta(t) \geq \frac{1}{2}$  and

$$\frac{d}{dt}\psi \leq \Delta\psi - 2\bar{\eta}^{-p}\nabla\bar{\eta}^p\nabla\psi - c_{17}\left\{e^{\frac{1}{2}c_{15}t}\psi^2 - e^{\frac{1}{4}c_{15}t}\psi^{\frac{1}{2}} - \psi\right\}, \quad (7.4)$$

for a constant  $c_{17} > 0$ .

**Claim:** When  $t \geq T_0$ ,  $\|B\|^2 \leq c_{19}e^{-\tau t}$  for some positive constants  $c_{19}$  and  $\tau$ .

We prove the claim. For any  $t_0 \in [T_0, +\infty)$ , we consider for each  $t \in [T_0, t_0]$  a point  $x_t$  such that  $\psi(x_t, t)$  attains its maximum  $\psi_{\max}(t)$ . Since  $\psi_{\max}$  is a locally Lipschitz function on  $[T_0, t_0]$ , we may take  $t_1 \in [T_0, t_0]$  a point where this maximum is achieved. If  $t_1 = T_0$ , we have done. Thus we may assume  $t_1 > T_0$ . At  $(x_{t_1}, t_1)$ ,  $\Delta\psi \leq 0$ ,  $\nabla\psi = 0$  and  $\frac{d}{dt}\psi \geq 0$ . Thus from (7.4) at  $(x_{t_1}, t_1)$ ,

$$e^{\frac{1}{2}c_{15}t}\psi^2 - e^{\frac{1}{4}c_{15}t}\psi^{\frac{1}{2}} - \psi \leq 0.$$

Let  $\bar{\psi}(x_{t_1}, t_1) = \sqrt{\psi(x_{t_1}, t_1)}$ . Then,  $e^{\frac{1}{2}c_{15}t_1}\bar{\psi}^3 - \bar{\psi} \leq e^{\frac{1}{4}c_{15}t_1}$ , what implies at  $(x_{t_1}, t_1)$

$$\bar{\psi}^3 \leq e^{\frac{1}{4}c_{15}t_1}\bar{\psi}^3 \leq 1 + e^{-\frac{1}{4}c_{15}t_1}\bar{\psi} \leq 1 + \bar{\psi}.$$

Thus, there is a constant  $c_{18} > 0$  that does not depend on  $t_0$ , such that  $\bar{\psi}(x_{t_1}, t_1) \leq c_{18}$ . Therefore,

$$\max_{M_{t_0}}\psi = \psi(x_{t_0}, t_0) \leq \psi(x_{t_1}, t_1) \leq c_{18}^2.$$

Then for a constant  $c_{19}$  not depending on  $t_0$  and  $t$ ,  $\max_{M_{t_0}}\|B\|^2 \leq c_{19}e^{-\frac{1}{2}c_{15}t_0}$ . Since  $t_0$  is arbitrary, we prove the claim.

By the claim, we have the estimate of  $\|H\|$

$$\|H\|^2 \leq \frac{c_{19}}{m}e^{-\tau t}. \quad (7.5)$$

By (7.5) and (6.1), we have  $\bar{d}(F_t(p), F_0(p)) \leq c_{20}$  for a positive constant  $c_{20}$ . Therefore  $F_t(p)$  lies in a compact region of  $\bar{M}$ . From the proof of corollary 6.1,  $F_{t_n}$  converges uniformly to a limit map  $F_\infty$  for a sequence  $t_n \rightarrow \infty$ . Moreover  $F_\infty$  is a spacelike graph of a map  $f_\infty$ . By (7.2)  $f_\infty = \text{constant}$ . Now we prove the limit is unique. Following the proof of corollary 6.1, in (6.2) we have in this case

$$d(F_t, F_\infty) \leq \left| \int_t^{t_n} \|H\|_{\max} ds \right| + d(F_{t_n}, F_\infty) \leq \frac{c_{19}\tau}{m}|e^{-\tau t} - e^{-\tau t_n}| + d(F_{t_n}, F_\infty)$$

and the right hand side converges to zero. This implies  $F_t$  to converge to  $F_\infty$  in  $C^\infty(\Sigma_1, \bar{M})$ . To see this, note that if there exists  $t_N \rightarrow +\infty$  such that  $d(F_{t_N}, F_\infty) \geq c_{30}$ ,  $c_{30}$  a positive constant, where  $d$  is the distance function relative to the  $C^1$ -Hölder norm, then, extracting a subsequence  $t_n$  of  $t_N$  with  $F_{t_n}$  converging in  $C^1(\Sigma_1, \bar{M})$  this sequence must converge to  $F_\infty$ , leading to a contradiction. By induction we prove  $F_t$  converges to  $F_\infty$  in  $C^k$ , for each  $k \geq 0$ .  $\square$

*Proof of theorem 1.2.* We consider the pseudo-Riemannian product space  $\bar{M} = \Sigma_1 \times \Sigma_2$  equipped with the pseudo-Riemannian metric  $g_1 - g'_2$ ,  $g'_2 = \rho^{-1}g_2$  for a constant  $\rho > 0$ . The spacelike condition for  $f$  means  $f^*g_2 < \rho g_1$ . The curvature tensor  $R'$  of  $g'_2$  satisfies  $R'_2(X', Y', X', Y') = \rho R_2(X, Y, X, Y)$ , where  $X' = \sqrt{\rho}X$  and  $Y' = \sqrt{\rho}Y$ . If  $K_1 > 0$ , then  $K_1 \geq K'_2$  is satisfied if we assume  $\rho \leq \min_{\Sigma_1} K_1 / \sup_{\Sigma_2} K_2^+$ . If  $K_2 \leq 0$  we may take  $\rho = +\infty$ . If  $\text{Ricci}_1 > 0$  and  $K_2 \leq -c < 0$ , we may take any  $\rho \geq \max_{\Sigma_1} K_1^- / c$ , where  $K^- = \max\{-K, 0\}$ , and in particular  $\rho = +\infty$ . Then we are in the conditions of theorem 1.1(3).  $\square$

*Proof of corollary 1.2.* Under the assumptions,  $\sum_i \lambda_i^2 + 1 \leq \Pi_i(1 + \lambda_i^2) < 2$ . In particular,  $\Gamma_f$  is a spacelike graph for the corresponding pseudo-Riemannian structure of  $\Sigma_1 \times \Sigma_2$ .  $\square$

*Proof of corollary 1.3.* We assume  $f$  minimizes the  $\phi$ -energy functional. Let  $f_t$  given by theorem 1.2. From (7.2) of lemma 7.1 and the assumptions on  $\phi$ , we have  $E_\phi(f) \leq \liminf E_\phi(f_t) = E_\phi(f_\infty) = 0$ , when  $t \rightarrow +\infty$ . Thus,  $E_\phi(f) = 0$ , and so  $f$  is constant.  $\square$

*Remark.* We note that when  $E_\phi$  is the usual energy functional of  $f$  corollary 1.3 can be obtained using a simple Weitzenböck formula. Since  $f$  is harmonic

$$\Delta \|df\|^2 = \|\nabla df\|^2 + \sum_{i \neq j} \lambda_i^2 (K_1(P_{ij}) - \lambda_j^2 K_2(P'_{ij}))$$

that under the curvature conditions of theorem 1.2,  $K_1 \geq K_2$  with  $K_1 \geq 0$ , or  $\text{Ricci}_1 \geq 0$  and  $K_2 \leq 0$ , we have  $\Delta \|df\|^2 \geq 0$ , what implies  $\|df\|$ , and so by the above equation  $f$  is totally geodesic, and if  $\text{Ricci}_1 > 0$  at some point, then  $\lambda_i = 0$ , that is  $f$  is constant. So, the corollary is mainly interesting for  $\phi$  not the square of the norm. But this argument also shows that the curvature condition  $\text{Ricci}_1 \geq 0$  with  $K_2 < 0$  is the expected one, since in this case, it is well known [7], that any map  $f : \Sigma_1 \rightarrow \Sigma_2$  is homotopic to an harmonic map, and so, under the condition  $\text{Ricci}_1 > 0$ , necessarily to a constant one.

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